Generalized Clapeyron's theorem

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Abstract

Clapeyron's Theorem in classical linear elasticity provides a way to explicitly express the energy stored in an equilibrium configuration in terms of the work of the forces applied on the boundary. We derive several new integral relations which can be viewed as nonlinear analogs of this classical result, reinterpreting them as rather general statements within Calculus of Variations. In the framework of nonlinear elasticity these relations reflect various partial symmetries of the material response, for instance, scale-invariance or scaling homogeneity. In particular, when the energy functional is scale-free, the obtained result can be interpreted as the Generalized Clapeyron's Theorem (GCT). Remarkably, it combines rather naturally the work of physical and configurational forces. We present a series of illuminating case studies showing the variety of applications of various obtained relations in different seemingly unrelated problems of mechanics.

1 Introduction

A well known theorem of linear elasticity, first attributed to Clapeyron in [63] states that the elastic energy stored in a body loaded by (dead) forces is equal to half of the work done by the loading device [67, 100]. More precisely, it states that

$$\int_{\Omega} W(\boldsymbol{x}, e(\boldsymbol{u}))d\boldsymbol{x} = \frac{1}{2}\int_{\partial\Omega} (\boldsymbol{\sigma}\boldsymbol{n}, \boldsymbol{u})dS(\boldsymbol{x}), \tag{1.1}$$

where $W(\boldsymbol{x}, \boldsymbol{\varepsilon}) = (1/2)(\mathsf{C}(\boldsymbol{x})\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})$, $\mathsf{C}(\boldsymbol{x})$ is the elasticity tensor , $\boldsymbol{u}(\boldsymbol{x})$ is the displacement, $\boldsymbol{\sigma}(\boldsymbol{x}) = \mathsf{C}e(\boldsymbol{u})$ is the linear stress tensor and

$$e(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u}(\boldsymbol{x}) + (\nabla\boldsymbol{u}(\boldsymbol{x}))^T) \tag{1.2}$$

is the linear strain tensor. While this theorem is obtained by simple integration by parts and is widely used in applications [11, 71, 49, 74] it carries a certain flavor of mystery.

First, it has never been published by the author himself [108, p. 418]. Second, it contains this puzzling disappearance of the half of the work, which can be linked to the fact that elastostatics

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is only a weak limit of elastodynamics (as the appropriately normalized magnitudes of the characteristic velocities diverge) [31]. Finally, to the best of our knowledge, a relation (1.1) has been viewed as strictly limited to classical elasticity in the sense that an adequate analog does not exist in geometrically and physically nonlinear elasticity.

Here we close the apparent gap between linear and nonlinear elasticity perspectives on surface representation of the bulk energy and expose a deeper structure of the classical Clapeyron's Theorem unifying it with some other similar results in mechanics and physics. This will not only allows us to generalize Clapeyron's Theorem beyond linear elasticity but will also open the possibility to view it as a general result in the broader Calculus of Variations.

We begin with the observation that the abstract problem which one can associate with the classical Clapeyron's Theorem is to find conditions ensuring that the energy of an extremal configuration can be expressed through the data on the boundary. Our solution of the problem links the generalization of Clapeyron's Theorem with scale invariance of the elastic energy, which we show to be sufficient alone for obtaining a representation of the total energy as a surface integral. Remarkably, while in the classical Clapeyron's Theorem the implied boundary term contains the work of physical forces only, we show that in our Genralized Clapeyron's Theorem (GCT), the energy of an extremal configuration can be expressed through what can be interpreted as the generalized work of both physical and configurational forces [15, 43, 85, 65, 103, 72]. This shows that GCT represents an indirect generalization of the classical Clapeyron's Theorem. A direct generalization of this classical result, that does not involve any reference to configurational forces, relies on what we call p-homogeneity of the energy density. We show that if both scale invariance and p-homogeneity are operative at the same time, there is a corresponding conserved quantity that links the work of physical and configurational forces. Eliminating one or the other from GCT using this link, we can express the equilibrium energy through the work of either only physical or only configurational forces.

The proposed broader reading of the Clapeyron's Theorem allows one to advance in several directions simultaneously, from generalizing the classical Green's formula in nonlinear elastostatics [39] (see also [47]), to obtaining a far reaching multidimensional generalization of the classical result of the Calculus of Variations expressing the minimal value of the energy through the Weierstrass excess function [58, 37, 36, 93].

The crucial step in our analysis is the realization that behind the conventional Clapeyron's Theorem is a formula first obtained by E. Noether [82, 83]. We recall, in this respect, that the well known Noether's Theorem [82] links the presence of variational symmetries with the existence of conservation laws in the form of divergence-free combinations of field variables. To obtain this result Noether considered the action of a continuous group of symmetries of the variational integral and showed that when a symmetry parameter is varied, the corresponding perturbation of the functional can be written as a sum of the Euler-Lagrange operator and a divergence [51, 83, 8]. The underlying Noether's formula has been used broadly to generate various integral identities in many different domains of science [113, 86, 87, 89, 90, 10, 88, 112, 56, 83]. Following some previous work, see for instance [10], we move away from symmetry groups and consider instead more general classes of transformations.

We begin by using Noether's formula to show that by means of an appropriate scaling of the dependent variable, classical Clapeyron's Theorem can be linked to degree 2 homogeneity of the quadratic energy density of linear elasticity. Our GCT then emerges from the same type of analysis but now using a scaling reparametrization transformation, involving both dependent and independent variables, which does not leave the functional invariant but multiplies it by a scalar factor.

As we have already mentioned, an intriguing feature of the GCT is that it naturally combines what is known in solid mechanics as the work performed by the Piola and Eshelby stresses [27, 97, 42, 84, 102, 44]. Moreover, we show that there exists a higher dimensional representation of GCT featuring a new tensor *combining* the Piola stress and the Eshelby stress. To obtain such a representation we first observe that for parametric problems energy density is a 1-homogeneous function of deformation gradient and therefore one can express the total energy along an extremal as a surface integral. Then we show that one can take a general non parametric problem and always map it into a higher dimensional parametric problem obtaining a desired boundary integral representation in the higher dimensional space. This higher dimensional representation allows one to give to the GCT a compact form involving the implied combination of the Piola and Eshelby stresses.

Since our work is motivated by applications which allow for surfaces of jump discontinuities of field gradients we account in our representation of GCT of inherent lack of smoothness of extremals [60, 59, 5, 36]. This issue becomes particularly important in our analysis of GCT applicable to nonlinear elastodynamics where the formation of singularities is unavoidable despite the presence of the overall variational structure [22, 33].

To highlight the utility of the obtained results, we present a series of illuminating case studies. In particular, we consider a generic problem of void formation in a nonlinear elastic body where the possibility of scaling reparametrization allows one to study infinitely small voids in a finite domain as finite voids in the infinite space and show that the use of GCT allows one to unify several previous partial results. We show further that GCT opens the way to addressing in a new way some important problems of material stability ranging from fracture to phase nucleation. We also provide evidence that another important set of applications of the GCT is associated with the possibility to rule out the presence of strong local minimizers that are not global in nonlinear elasticity in the case of hard device loading.

Finally, we mention again that since partial invarinace of the type studied in this paper is a rather general property of variational problems, the obtained nonlinear versions of the classical Clapeyron's Theorem applies to a broad class of variational problems not at all related to elasticity theory. For instance, our methods offer a unified approach to Rellich and Pohozaev type identities normally used in reaction diffusion framework. In this perspective, the obtained integral identities should be viewed as general results within Calculus of Variations reflecting particular structure of the corresponding energy functionals. Notwithstanding the immense breadth of possible application of GCT in different disciplines, we have chosen to focus our presentation on problems of solid mechanics while dedicating only brief comments to other applications.

The paper is organized as follows. Our Section 2 contains the background material including the introduction of inner and outer variations, the definitions of Piola and Eshelby stresses, the analysis of general transformation families for variational problems and the derivation of the generalized Noether's formula, which we illustrate in a detailed study of multidimensional elastodynamics.

In Section3 we specialize the generalized Noether's formula to the case of partial symmetries with parametric Lagrangians as a prominant example. We then show how different types of partial symmetries allow one to use the general Noether's formula to derive various invariant integrals, for instance, the nonlinear versions of J, L and M integrals, whose linearized analogs have long been known in elasticity theory. As a case study, we present in this section a derivation of a generalized version of Pohozaev's identities and show how to use them to prove uniqueness in a broad class of nonlinear boundary value problems.

The actual GCT is derived only in Section 4 where we introduce the idea of scale invariance and derive the integral identity resulting in the corresponding generalized Noether's formula. We then map an original non parametric problem into a higher dimensional parametric one and take advantage of the ensuing scale invariance to obtain a hiher dimensional version of the GCT in the extended space, naturally combining the Piola and Eshelby stresses. In the same section we present versions of GCT applicable in the presence of constraints and in the case when extremals exhibit surfaces of jump discontinuities of field gradients. A special place among what we call "partial symmetries" is occupied by p-homogeneity which we use, for instance, to simplify GCT in the case of linear elasticity where the energy density is a quadratic function of the strain tensor. The case study in this section is again the elastodynamics where now we focus on the derivation of a version of GCT that accounts for the emergence of shock waves.

Our Section 5 is devoted to various applications of the obtained results. In particular we show how the use of GCT allows one to formulate a new necessary condition of metastability, establish the non-existence of strong local minima, and gain nontrivial insight into notoriously inaccessible quasi-convex envelopes of elastic energies. We then use GCT to obtain a new general representation of the energy release associated with the void formation in nonlinear elasticity theory. In particular, we provide a novel resolution of the well known "Griffith's error" [99, 28, 96, 53]. The corresponding linear elastic problem is discussed in full detail as a case study.

Our conclusions are summarized in Section 6. Several results of purely technical nature are presented in the form of Appendixes in Section A.

2 Preliminaries

We begin by introducing a general variational problem of minimizing the energy functional

$$E[\boldsymbol{y}] = \int_{\Omega} W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})) d\boldsymbol{x}, \tag{2.1}$$

where, using the language of elasticity theory, we interpret the function $\boldsymbol{y} : \Omega \to \mathbb{R}^m$ as a deformation field defined on the reference (Lagrangian) configuration $\Omega \subset \mathbb{R}^n$. With elastodynamics and other applications in mind, we deal here with a more general framework where m and n are arbitrary, the energy density W is allowed to depend on both $\boldsymbol{x}$ and $\boldsymbol{y}$, and most importantly, where $\boldsymbol{y}(\boldsymbol{x})$ is assumed to be only Lipschitz continuous, most often with only singularities of $\nabla \boldsymbol{y}(\boldsymbol{x})$ being smooth surfaces of jump discontinuity, as occurs in the presence of shocks and during martensitic phase transitions. Since there are many examples of Lipschitz extremals with more general singular sets, even for strictly convex energy densities [23, 78, 4, 98, 105, 77], in each case we will specify the extra regularity assumptions placed on $\boldsymbol{y}(\boldsymbol{x})$, which will otherwise always be assumed Lipschitz continuous.

The image $\Omega^* = \boldsymbol{y}(\Omega)$ is called actual or Eulerian configuration. In classical nonlinear elasticity it is assumed that $W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}) = W(\boldsymbol{F})$ does not depend explicitly on $\boldsymbol{x}$ and $\boldsymbol{y}$ and is objective, i.e. $W(\boldsymbol{RF}) = W(\boldsymbol{F})$ for all $\boldsymbol{R} \in SO(3)$ and all $\boldsymbol{F}$, $\det \boldsymbol{F} > 0$. However, here we are not making these assumptions, as our conclusions hold for a far more general class of energy densities.

We will assume, in general, that W is of class C^1, if not globally, then on the open set containing the range of $(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x}))$ for any configuration $\boldsymbol{y}$ under consideration. In the context of three-dimensional nonlinear elasticity, for example, this means that the range of $\nabla \boldsymbol{y}(\boldsymbol{x})$ will be a closed subset of $\{\boldsymbol{F} \in \mathbb{R}^{3\times 3} : \det \boldsymbol{F} > 0\}$, eliminating the difficulties typically arising when one considers Sobolev solutions, whose existence is guaranteed by [3]. In some situations that we will identify explicitly, the regularity in the $\boldsymbol{x}$ variable will not be assumed, as in the case of a heterogenous or composite media.

2.1 First variation of a graph

We begin with a brief derivation of the first variation of the energy. Following Noether, we adopt a more geometric point of view and regard the energy density W as a function on the tangent bundle $T\Gamma$, where

$$\Gamma = \{(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})) : \boldsymbol{x} \in \Omega\} \subset \mathbb{R}^{m+n}$$

is the graph of $\boldsymbol{y}(\boldsymbol{x})$. We will now examine the effect of the graph perturbation on the energy functional. Hence, we consider a smooth family of Lipschitz maps $\boldsymbol{\Phi}_\epsilon : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$, such that $\boldsymbol{\Phi}_0(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y})$. When ϵ is sufficiently small, $\Gamma_\epsilon = \boldsymbol{\Phi}_\epsilon(\Gamma)$ would still be a graph of some Lipschitz function $\widetilde{\boldsymbol{y}}_\epsilon$, and our goal is to derive the formula for the first variation

$$\delta E = \left.\frac{dE[T\Gamma_\epsilon]}{d\epsilon}\right|_{\epsilon=0}, \tag{2.2}$$

where

$$E[T\Gamma_\epsilon] = \int_{\Omega_\epsilon} W(\widetilde{\boldsymbol{x}}, \boldsymbol{y}_\epsilon(\widetilde{\boldsymbol{x}}), \nabla \boldsymbol{y}_\epsilon(\widetilde{\boldsymbol{x}}))d\widetilde{\boldsymbol{x}}. \tag{2.3}$$

To explain the notations in (2.3), we write

$$\boldsymbol{\Phi}_\epsilon(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}, \epsilon), \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{y}, \epsilon)), \tag{2.4}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are Lipschitz functions of $(\boldsymbol{x}, \boldsymbol{y})$ and of class C^1 in ϵ, such that

$$\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}, 0) = \boldsymbol{x}, \quad \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{y}, 0) = \boldsymbol{y}.$$

According to (2.4) $\widetilde{\boldsymbol{y}}_\epsilon(\widetilde{\boldsymbol{x}}) = \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon)$, in other words,

$$\boldsymbol{y}_\epsilon(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon)) = \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon), \tag{2.5}$$

and hence, we obtain (2.3), where Ω_ϵ is the image of Ω under the transformation $\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon)$.

To compute δE given by (2.2) we first introduce the notations

$$\delta\boldsymbol{x}(\boldsymbol{x}, \boldsymbol{y}) = \left.\frac{\partial \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}, \quad \delta\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{y}) = \left.\frac{\partial \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{y}, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0},$$

even though in our subsequent derivations we will only use the Lipschitz functions

$$\delta\boldsymbol{x}(\boldsymbol{x}) = \delta\boldsymbol{x}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})), \qquad \delta\boldsymbol{y}(\boldsymbol{x}) = \delta\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})).$$

We then make in (2.3) the change of variables $\widetilde{\boldsymbol{x}} = \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon)$, and obtain

$$E[T\Gamma_\epsilon] = \int_\Omega W(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon), \boldsymbol{y}_\epsilon(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon)), \nabla\boldsymbol{y}_\epsilon(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon)))\det(\nabla\boldsymbol{X})d\boldsymbol{x},$$

where $\nabla\boldsymbol{X} = \nabla_{\boldsymbol{x}}(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon))$. Differentiating (2.5) in $\boldsymbol{x}$ we obtain

$$\nabla\boldsymbol{y}_\epsilon(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon))\nabla\boldsymbol{X} = \nabla\boldsymbol{Y}$$

Therefore,

$$E[T\Gamma_\epsilon] = \int_\Omega W(\boldsymbol{X}, \boldsymbol{Y}, \nabla\boldsymbol{Y}(\nabla\boldsymbol{X})^{-1})\det(\nabla\boldsymbol{X})d\boldsymbol{x}, \tag{2.6}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are evaluated at $(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \epsilon)$. We note that

$$\left.\frac{\partial\nabla\boldsymbol{X}}{\partial\epsilon}\right|_{\epsilon=0} = \nabla_{\boldsymbol{x}}\delta\boldsymbol{x}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})), \quad \left.\frac{\partial\nabla\boldsymbol{Y}}{\partial\epsilon}\right|_{\epsilon=0} = \nabla_{\boldsymbol{x}}\delta\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})).$$

Now, we can differentiate (2.6) in ϵ at $\epsilon = 0$ and obtain

$$\delta E = \int_\Omega \{W_{\boldsymbol{x}} \cdot \delta\boldsymbol{x} + W_{\boldsymbol{y}} \cdot \delta\boldsymbol{y} + \langle W_{\boldsymbol{F}}, \nabla\delta\boldsymbol{y} - \nabla\boldsymbol{y}\nabla\delta\boldsymbol{x}\rangle + W\nabla \cdot \delta\boldsymbol{x}\}d\boldsymbol{x},$$

where $\delta\boldsymbol{x} = \delta\boldsymbol{x}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}))$ and $\delta\boldsymbol{y} = \delta\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}))$.

We now introduce important notations which will be essentially used in what follows. First, we define the tensor field

$$\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}) = W_{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}), \tag{2.7}$$

to which we refer as the *Piola stress tensor* using its narrow meaning in classical elasticity theory; in a general field theory it is known as canonical momentum or current tensor. Second, we define another tensor field

$$\boldsymbol{P}^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}) = W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})\boldsymbol{I}_n - \boldsymbol{F}^T\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}), \tag{2.8}$$

and refer to it as the *Eshelby tensor*, again, referring to a more limited notion used in nonlinear elasticity; in a general field theory this object is known as the energy momentum tensor. The corresponding background material, explaining the origin of these definitions in more detail, can be found in the monographs [97, 43, 44, 72, 34, 62, 95].

Using these definitions we can rewrite the formula for δE as follows

$$\delta E = \int_\Omega \{W_{\boldsymbol{y}} \cdot \delta\boldsymbol{y} + W_{\boldsymbol{x}} \cdot \delta\boldsymbol{x} + \langle \boldsymbol{P}, \nabla\delta\boldsymbol{y}\rangle + \langle \boldsymbol{P}^*, \nabla\delta\boldsymbol{x}\rangle\}d\boldsymbol{x}, \tag{2.9}$$

where $\boldsymbol{P}$ and $\boldsymbol{P}^*$ are evaluated at $(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla\boldsymbol{y}(\boldsymbol{x}))$. If all the functions above, W, $\boldsymbol{y}$, $\boldsymbol{X}$, and $\boldsymbol{Y}$ are of class C^2, we can integrate by parts and rewite (2.9) as

$$\delta E = \int_\Omega \{\mathfrak{E}_W(\boldsymbol{x}) \cdot \delta\boldsymbol{y} + \mathfrak{E}_W^*(\boldsymbol{x}) \cdot \delta\boldsymbol{x}\}d\boldsymbol{x} + \int_{\partial\Omega} \{\boldsymbol{P}\boldsymbol{n} \cdot \delta\boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n} \cdot \delta\boldsymbol{x}\}d\boldsymbol{x}, \tag{2.10}$$

where

$$\mathfrak{E}_W(\boldsymbol{x}) = W_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})) - \nabla \cdot \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}), \tag{2.11}$$

and

$$\mathfrak{E}^*_W(\boldsymbol{x}) = W_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})) - \nabla \cdot \boldsymbol{P}^*(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})). \tag{2.12}$$

The fields $\mathfrak{E}_W(\boldsymbol{x})$ and $\mathfrak{E}^*_W(\boldsymbol{x})$ are linked through a relation derived originally by Noether [82]:

$$\mathfrak{E}^*_W(\boldsymbol{x}) = -(\nabla \boldsymbol{y})^T \mathfrak{E}_W(\boldsymbol{x}), \tag{2.13}$$

Indeed,

$$\begin{aligned}\mathfrak{E}^*_W(\boldsymbol{x})_\alpha = \frac{\partial W}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha}[W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y})] + \frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta} P^\beta_i + \frac{\partial y^i}{\partial x^\alpha}\frac{\partial P^\beta_i}{\partial x^\beta} = \\ \frac{\partial W}{\partial x^\alpha} - \frac{\partial W}{\partial x^\alpha} - \frac{\partial W}{\partial y^i}\frac{\partial y^i}{\partial x^\alpha} - P^\beta_i \frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta} P^\beta_i + \frac{\partial y^i}{\partial x^\alpha}\frac{\partial P^\beta_i}{\partial x^\beta} = \\ -\frac{\partial W}{\partial y^i}\frac{\partial y^i}{\partial x^\alpha} + \frac{\partial y^i}{\partial x^\alpha}\frac{\partial P^\beta_i}{\partial x^\beta} = -\frac{\partial y^i}{\partial x^\alpha}\mathfrak{E}_W(\boldsymbol{x})_i,\end{aligned}$$

which is (2.13) written in components.

The massive cancellations of terms in the above derivation of (2.13) suggests that there is a deeper underlying reason for (2.13) to hold. We will therefore rederive it as a corollary of formula (2.9), valid for any Lipschitz $\overline{\boldsymbol{y}} \in W^{2,1}_{\text{loc}}(\Omega; \mathbb{R}^m)$. We present the result in the form of the theorem:

THEOREM **2.1.** *Suppose W is of class C^1, and $\overline{\boldsymbol{y}} \in W^{2,1}_{\text{loc}}(\Omega; \mathbb{R}^m) \cap W^{1,\infty}(\Omega; \mathbb{R}^m)$. Then (2.13) holds in the sense of equality of distributions, where $(\nabla \boldsymbol{y})^T \mathfrak{E}_W$ is understood as the distribution*

$$((\nabla \boldsymbol{y})^T \mathfrak{E}_W)_i = \frac{\partial \overline{y}^k}{\partial x^i} W_{y^k} - \frac{\partial}{\partial x^j}\left[\frac{\partial \overline{y}^k}{\partial x^i} P^j_k\right] + \frac{\partial^2 \overline{y}^k}{\partial x^i \partial x^j} P^j_k,$$

which makes sense, since our regularity assumptions on $\overline{\boldsymbol{y}}$ imply that the third term above is in $L^1_{\text{loc}}(\Omega)$.

Proof. The idea is to construct a sufficiently large family of nontrivial deformations of the $(\boldsymbol{x}, \boldsymbol{y})$-space none of whose members change the graph of $\overline{\boldsymbol{y}}(\boldsymbol{x})$. Let $\boldsymbol{\phi} \in C^\infty_0(\Omega; \mathbb{R}^m)$ be arbitrary. Then $\boldsymbol{X}(\boldsymbol{x}, \epsilon) = \boldsymbol{x} + \epsilon \boldsymbol{\phi}(\boldsymbol{x})$ is a diffeomorphism of Ω onto itself for sufficiently small $|\epsilon|$, so that $\Omega_\epsilon = \Omega$. Then the deformation

$$\boldsymbol{\Phi}_\epsilon(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{X}(\boldsymbol{x}, \epsilon), \boldsymbol{y} + \overline{\boldsymbol{y}}(\boldsymbol{X}(\boldsymbol{x}, \epsilon)) - \overline{\boldsymbol{y}}(\boldsymbol{x})),$$

leaves the graph of $\overline{\boldsymbol{y}}(\boldsymbol{x})$ invariant. Indeed,

$$\boldsymbol{\Phi}_\epsilon(\boldsymbol{x}, \overline{\boldsymbol{y}}(\boldsymbol{x})) = (\boldsymbol{X}(\boldsymbol{x}, \epsilon), \overline{\boldsymbol{y}}(\boldsymbol{x}) + \overline{\boldsymbol{y}}(\boldsymbol{X}(\boldsymbol{x}, \epsilon)) - \overline{\boldsymbol{y}}(\boldsymbol{x})) = (\boldsymbol{X}(\boldsymbol{x}, \epsilon), \overline{\boldsymbol{y}}(\boldsymbol{X}(\boldsymbol{x}, \epsilon))).$$

Since $\boldsymbol{X}(\boldsymbol{x}, \epsilon)$ is a diffeomorphism of Ω, we conclude that $\boldsymbol{\Phi}_\epsilon(\Gamma) = \Gamma$, where Γ is the graph of $\overline{\boldsymbol{y}}(\boldsymbol{x})$. Thus, $\delta E = 0$ on the left-hand side of (2.9). We also compute

$$\delta \boldsymbol{x} = \boldsymbol{\phi}, \quad \delta \boldsymbol{y} = (\nabla \overline{\boldsymbol{y}})\boldsymbol{\phi}.$$

In that case formula (2.9) reads

$$0=\int_\Omega\{W_{\boldsymbol{x}}\cdot\boldsymbol{\phi}+\langle\boldsymbol{P}^*,\nabla\boldsymbol{\phi}\rangle+W_{\boldsymbol{y}}\cdot(\nabla\overline{\boldsymbol{y}})\boldsymbol{\phi}+\langle\boldsymbol{P},\nabla((\nabla\overline{\boldsymbol{y}})\boldsymbol{\phi})\rangle\}d\boldsymbol{x},\tag{2.14}$$

writing

$$\langle\boldsymbol{P},\nabla((\nabla\overline{\boldsymbol{y}})\boldsymbol{\phi})\rangle=P_k^j\frac{\partial\overline{y}^k}{\partial x^i}\frac{\partial\phi^i}{\partial x^j}+P_k^j\frac{\partial^2\overline{y}^k}{\partial x^i\partial x^j}\phi^i,$$

we see that (2.14) can be written as $0=\langle\mathfrak{E}_W^*|\boldsymbol{\phi}\rangle+\langle(\nabla\boldsymbol{y})^T\mathfrak{E}_W|\boldsymbol{\phi}\rangle$, where the angular bracket notation denotes the action of distributions on test functions. □

Remark 2.2. *When $\overline{\boldsymbol{y}}$ is merely Lipschitz continuous, the proof of Theorem 2.1 breaks down, since in this case $\nabla\delta\boldsymbol{y}$ is not an $L^1_{\rm loc}$ function and the derivation of formula (2.9) becomes invalid. Moreover, formula (2.9) itself does not make sense for a distributional $\nabla\delta\boldsymbol{y}$. Nonetheless, Theorem 2.1 can still be used on subdomains of Ω, where $\overline{\boldsymbol{y}}$ is of class $W^{2,1}$.*

Using this remark, we can apply the Noether formula (2.10) to the functional

$$E[\boldsymbol{y}]=\int_\Omega W(\nabla\boldsymbol{y})d\boldsymbol{x}\tag{2.15}$$

where the field $\boldsymbol{y}(\boldsymbol{x})$ is assumed to be of class C^2, outside of a smooth surface Σ of jump discontinuity of $\nabla\boldsymbol{y}$ where the distributional parts of the Lagrangian quantities $\mathfrak{E}_W$ and $\mathfrak{E}_W^*$ supported on the surface Σ must be taken into account. The identity (2.9) can be then rewritten in the form similar to (2.10)

$$\delta E=-\int_\Omega\{(\nabla\cdot\boldsymbol{P})_{\rm reg}\cdot\delta\boldsymbol{y}+(\nabla\cdot\boldsymbol{P}^*)_{\rm reg}\cdot\delta\boldsymbol{x}\}d\boldsymbol{x}$$
$$+\int_{\partial\Omega}\{\boldsymbol{P}\boldsymbol{n}\cdot\delta\boldsymbol{y}+\boldsymbol{P}^*\boldsymbol{n}\cdot\delta\boldsymbol{x}\}dS-\int_\Sigma\{[\![\boldsymbol{P}]\!]\boldsymbol{n}\cdot\delta\boldsymbol{y}+[\![\boldsymbol{P}^*]\!]\boldsymbol{n}\cdot\delta\boldsymbol{x}\}dS.\tag{2.16}$$

where $(\nabla\cdot\boldsymbol{P})_{\rm reg}$ and $(\nabla\cdot\boldsymbol{P}^*)_{\rm reg}$ are C^1 functions on $\Omega\setminus S$, suffering a jump discontinuity across Σ, defined by $\nabla\cdot\boldsymbol{P}(\boldsymbol{x})$ and $\nabla\cdot\boldsymbol{P}^*(\boldsymbol{x})$, $\boldsymbol{x}\in\Omega\setminus\Sigma$. If we follow the standard convention that the unit normal to a surface Σ always points from the "$-$" side of Σ to its "$+$" side, then

$$[\![\boldsymbol{P}]\!]=\boldsymbol{P}_+(\boldsymbol{x})-\boldsymbol{P}_-(\boldsymbol{x}),\quad \boldsymbol{x}\in\Sigma.$$

2.2 Extremal and stationary configurations

Our primary interest lies in problems of Calculus of Variations arising from elastostatics and elastodynamics. The latter is governed by the action principle, whereby

$$\delta E=0\tag{2.17}$$

for all outer variations $\delta\boldsymbol{y}\in C_0^\infty(\Omega;\mathbb{R}^m)$, and $\delta\boldsymbol{x}=0$; the former—by the energy minimization procedure, one of whose consequences is (2.17) for all variations $\delta\boldsymbol{y}\in C_0^\infty(\Omega;\mathbb{R}^m)$ and $\delta\boldsymbol{x}\in C_0^\infty(\Omega;\mathbb{R}^n)$. In both cases condition (2.17) requires that

$$\mathfrak{E}_W(\boldsymbol{x})=0,\tag{2.18}$$

which serves in such theories as the main governing equation.

Definition 2.3. *We say that the Lipschitz configuration $\boldsymbol{y}(\boldsymbol{x})$ is an extremal, if it satisfies the Euler-Lagrange equation (2.18) in the sense of distributions.*

This definition allows for singularities of $\nabla\boldsymbol{y}$ corresponding, for instance, to shock waves and phase boundaries [109]. We recall that stationarity is fully compatible with jump discontinuities of this type as it places restrictions on such singularities but does not rule them out. This issue is discussed in more detail in Section 4.4.

An important implication of Thorem 2.1 is that for any Lipschitz extremal, which is of class $W^{2,1}_{\rm loc}$ outside of a closed nowhere dense singular set S, $\mathfrak{E}^*_W(\boldsymbol{x})$ is a distribution supported on S. In the special case when $\nabla\boldsymbol{y}(\boldsymbol{x})$ suffers a jump discontinuity across a smooth surface $\Sigma\subset\Omega$, outside of which $\boldsymbol{y}$ is of class C^2, the distribution $\mathfrak{E}^*_W(\boldsymbol{x})$ has a particularly simple form. To derive it, we take an arbitrary point $\boldsymbol{x}_0\in\Sigma$, choose a direction of the normal $\boldsymbol{n}_0$ to Σ at $\boldsymbol{x}_0$ and a radius $r>0$, such that Σ splits $B(\boldsymbol{x}_0,r)$ into two disjoint subdomains $B^+(\boldsymbol{x}_0,r)$ and $B^-(\boldsymbol{x}_0,r)$, with $\boldsymbol{n}_0$ pointing from $B^-(\boldsymbol{x}_0,r)$ and into $B^+(\boldsymbol{x}_0,r)$, by our convention. Then there is a unique choice of the smooth normal $\boldsymbol{n}(\boldsymbol{x})$ to Σ at $\Sigma\cap B(x_0,r)$, such that $\boldsymbol{n}(\boldsymbol{x}_0)=\boldsymbol{n}_0$. Then for any test function $\boldsymbol{\phi}\in C_0^\infty(B(\boldsymbol{x}_0,r))$,

$$\langle\mathfrak{E}^*_W(\boldsymbol{x}),\boldsymbol{\phi}\rangle=\int_{B(\boldsymbol{x}_0,r)}\{W_{\boldsymbol{x}}\cdot\boldsymbol{\phi}+\langle\boldsymbol{P}^*,\nabla\boldsymbol{\phi}\rangle\}d\boldsymbol{x}=$$
$$\int_{B^+(\boldsymbol{x}_0,r)}\{W_{\boldsymbol{x}}\cdot\boldsymbol{\phi}+\langle\boldsymbol{P}^*,\nabla\boldsymbol{\phi}\rangle\}d\boldsymbol{x}+\int_{B^-(\boldsymbol{x}_0,r)}\{W_{\boldsymbol{x}}\cdot\boldsymbol{\phi}+\langle\boldsymbol{P}^*,\nabla\boldsymbol{\phi}\rangle\}d\boldsymbol{x}.$$

We can now integrate by parts on each of the domains $B^\pm(\boldsymbol{x}_0,r)$, taking (2.13) and $\mathfrak{E}_W(\boldsymbol{x})=0$ into account:

$$\langle\mathfrak{E}^*_W(\boldsymbol{x}),\boldsymbol{\phi}\rangle=-\int_\Sigma[\![\boldsymbol{P}^*]\!]\boldsymbol{n}\cdot\boldsymbol{\phi}\,dS. \tag{2.19}$$

The vector field $[\![\boldsymbol{P}]\!]^*\boldsymbol{n}$ in (2.19) can be simplified further, taking into account again that $\mathfrak{E}_W(\boldsymbol{x})=0$, which implies that

$$[\![\boldsymbol{P}]\!]\boldsymbol{n}=\boldsymbol{0}. \tag{2.20}$$

In this case we can write

$$[\![\boldsymbol{P}^*]\!]\boldsymbol{n}=[\![W]\!]\boldsymbol{n}-[\![\boldsymbol{F}^T\boldsymbol{P}]\!]\boldsymbol{n}=[\![W]\!]\boldsymbol{n}-[\![\boldsymbol{F}]\!]^T\boldsymbol{P}\boldsymbol{n},$$

where we use the usual notation $\boldsymbol{F}(\boldsymbol{x})=\nabla\boldsymbol{y}(\boldsymbol{x})$. We now recall that on the smooth surface of jump discontinuity of $\nabla\boldsymbol{y}$ the following Hadamard relations hold:

$$[\![\boldsymbol{F}]\!]=\boldsymbol{a}\otimes\boldsymbol{n}, \tag{2.21}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal on an $n-1$-dimensional surface $\Sigma\subset\mathbb{R}^n$ and $\boldsymbol{a}:\Sigma\to\mathbb{R}^m$ is a C^1 vector field on Σ. Then

$$[\![\boldsymbol{P}^*]\!]\boldsymbol{n}=([\![W]\!]-\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{a})\boldsymbol{n}=([\![W]\!]-\langle\boldsymbol{P}_\pm,[\![\boldsymbol{F}]\!]\rangle)\boldsymbol{n}=p^*\boldsymbol{n}.$$

Hence,

$$\mathfrak{E}^*_W(\boldsymbol{x})=-p^*_\Sigma\boldsymbol{n}\delta_\Sigma, \tag{2.22}$$

where δ_Σ is a distribution defined by

$$\langle \delta_\Sigma, \phi \rangle = \int_\Sigma \phi(\boldsymbol{x}) dS, \quad \forall \phi \in C_0^\infty(\Omega).$$

In (2.22) we also introduced a continuous scalar field

$$p_\Sigma^* = [\![W]\!] - \langle \boldsymbol{P}_\pm, [\![\boldsymbol{F}]\!] \rangle \tag{2.23}$$

on the smooth surface of jump discontinuity Σ. Here $\boldsymbol{P}_\pm$ indicates that it does not matter whether one uses $\boldsymbol{P}_+$ or $\boldsymbol{P}_-$ in the formula for p_Σ^*, as long as one uses this formula for an extremal, so that (2.20) holds.

As we have already mentioned, in static problems we are usually interested in minima of energy functionals, subject to specified boundary conditions. Then, Lipschitz minimizers of (2.1) must satisfy $\delta E = 0$ for all variations $\delta \boldsymbol{x} \in C_0^\infty(\Omega; \mathbb{R}^n)$, and $\delta \boldsymbol{y} \in C_0^\infty(\Omega; \mathbb{R}^m)$. Hence, in addition to (2.18) the solutions of such problems must also satisfy

$$\mathfrak{E}_W^*(\boldsymbol{x}) = 0, \tag{2.24}$$

again understood in the sense of distributions.

Definition 2.4. *We say that a Lipschitz configuration $\boldsymbol{y}(\boldsymbol{x})$ is* stationary *if it is an extremal in the sense of Definition 2.3, and additionally satisfies (2.24).*

Note that in view of Theorem 2.1 if $\overline{\boldsymbol{y}} \in W_{\text{loc}}^{2,1}(\Omega; \mathbb{R}^m) \cap W^{1,\infty}(\Omega; \mathbb{R}^m)$ is an extremal, then it is stationary in the sense of Definition 2.4. Indeed $\nabla \overline{\boldsymbol{y}} \boldsymbol{\phi} \in W_0^{1,1}(\Omega; \mathbb{R}^m)$ and

$$\int_\Omega \{W_{\boldsymbol{y}} \cdot \boldsymbol{\psi} + \langle \boldsymbol{P}, \nabla \boldsymbol{\psi} \rangle\} d\boldsymbol{x} = 0, \quad \forall \boldsymbol{\psi} \in C_0^\infty(\Omega; \mathbb{R}^m).$$

By the density argument it follows that

$$\int_\Omega \{W_{\boldsymbol{y}} \cdot \boldsymbol{\psi} + \langle \boldsymbol{P}, \nabla \boldsymbol{\psi} \rangle\} d\boldsymbol{x} = 0, \quad \forall \boldsymbol{\psi} \in W_0^{1,1}(\Omega; \mathbb{R}^m).$$

Taking $\boldsymbol{\psi} = \nabla \overline{\boldsymbol{y}} \boldsymbol{\phi}$ we obtain

$$\langle (\nabla \overline{\boldsymbol{y}})^T \mathfrak{E}_W | \boldsymbol{\phi} \rangle = 0 \quad \forall \boldsymbol{\phi} \in C_0^\infty(\Omega; \mathbb{R}^n).$$

Theorem 2.1 then implies that (2.24) holds.

Note also that formula (2.22) implies that for stationary extremals with a smooth surface of jump discontinuity Σ we must have

$$p_\Sigma^* = [\![W]\!] - \langle \boldsymbol{P}_\pm, [\![\boldsymbol{F}]\!] \rangle = 0. \tag{2.25}$$

In this sense stationarity, is not a consequence of equilibrium even for piecewise smooth $\boldsymbol{y}(\boldsymbol{x})$.

While we introduced in (2.7) and (2.8) two important tensorial fields, $\boldsymbol{P}$ and $\boldsymbol{P}^*$, which appear for instance in (2.10), their physical meaning has not been explained. Observe first that while Piola stress $\boldsymbol{P}$ acts on the virtual displacements $\delta \boldsymbol{y}$ of the points on the boundary of the domain

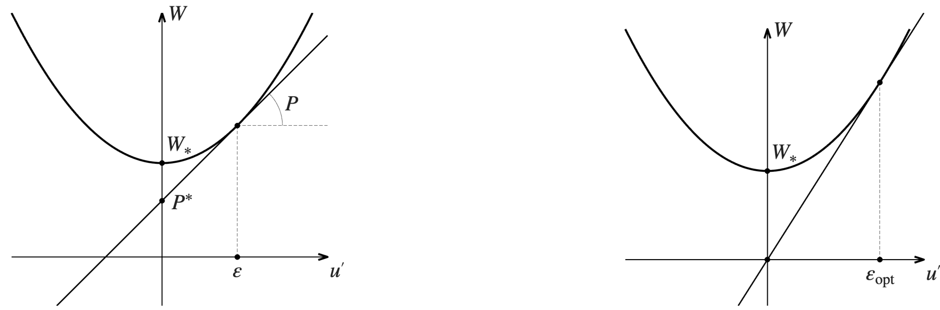

Figure 1: Relations between physical stress P, strain ε and the configurational stress P^* in equilibrium.

in the actual (Eulerian) space, the Eshely stress $\boldsymbol{P}^*$ acts on the virtual changes of the shape of the domain Ω in the reference (Lagrangian) space. The former is usually interpreted as the action of "physical forces", and the latter – as the action of "configurational forces". Therefore, we can conclude that the two terms in the surface integral in (2.10) represent the work of both physical and configurational forces, see for instance, [44]. More specifically, the implied two energy increments can be associated with physical stress, respectively configurational prestress. Here, while the meaning of the physical stress does not require any clarifications as it is related to the work of applied forces, we associate configurational prestress with the work that has been stored during the *creation* of the physical body. To elucidate the implied fundamental difference between $\boldsymbol{P}$ and $\boldsymbol{P}^*$ we present below to simple illustrative examples.

Example 2.5.

Consider a one-dimensional body, given in Lagrangian coordinates as an interval $[0, L]$. The stored energy is

$$E[u] = \int_0^L W(u')dx, \tag{2.26}$$

where $u(x)$ is the displacement field. We assume that the energy density function $W(\varepsilon)$ is a smooth, even, strictly convex function, such that $W'(0) = 0$, corresponding to the absence of the physical prestress. The configurational prestress is encoded by the residual energy $W_* = W(0) > 0$. The relationship between the physical (Piola) stress

$$P = W'(\varepsilon) \tag{2.27}$$

and the configurational (Eshelby) stress

$$P^* = W(\varepsilon) - \varepsilon W'(\varepsilon), \tag{2.28}$$

where the strain is $\varepsilon = u'(x)$ is shown in the left panel of Fig. 1. Behind this figure is the observation that in equilibrium we always have $(W'(\varepsilon))' = 0$ and due to the strict monotonicity of $W'(\varepsilon)$, expressing the strict convexity of $W(\varepsilon)$, all equilibrium configurations are the ones

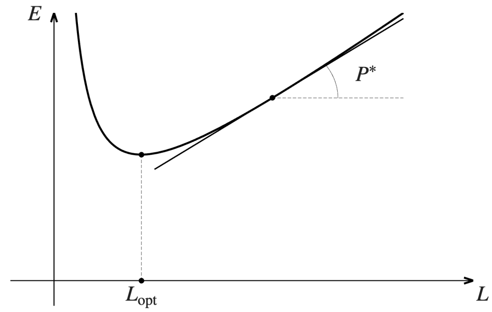

Figure 2: Energy as a function of the domain shape, when boundary displacements are fixed.

where ε is constant on $[0, L]$. Note that while P^* has the meaning of energy, P is a derivative of energy with respect to ε.

Consider now a slightly different situation, where the deformation of points on $\partial\Omega$ is prescribed, but the domain Ω itself is not fixed. In our 1D setting we can then consider that the numbers $U_0 = u(0)$ and $U_1 = u(L)$ are prescribed, while L can vary.

From a general point of view, in that case the admissible outer variations $\delta \boldsymbol{y}$ in (2.10) must vanish on $\partial\Omega$, but the boundary values of admissible inner variations $\delta \boldsymbol{x}$ are no longer restricted in any way. Then the vanishing of the first variation δE of the energy is equivalent to the Euler-Lagrange equation (2.18) and the Noether equation (2.24) augmented with the displacement boundary conditions together with the condition

$$\boldsymbol{P}^* \boldsymbol{n} = 0 \text{ on } \partial\Omega. \tag{2.29}$$

This leads to a free boundary problem, where the shape and size of Ω must be determined, in addition to the deformation $\boldsymbol{y}(\boldsymbol{x})$.

In our one dimensional example we can conclude that the energy extremum as a function of the shape/size of the domain is attained when $P^* = 0$. The right panel in Fig. 1 shows how the energy density function $W(\varepsilon)$ determines the value of ε_{opt} of the optimal strain. This gives us the value

$$L_{\text{opt}} = \frac{U_1 - U_0}{\varepsilon_{\text{opt}}}.$$

If U_0 and U_1 are prescribed, then, by strict convexity of $W(\varepsilon)$, there is a unique value of the constant equilibrium strain

$$\varepsilon = \frac{U_1 - U_0}{L}.$$

Thus, the energy as a function of L is

$$E = LW\left(\frac{U_1 - U_0}{L}\right).$$

It is easy to see that

$$\frac{dE}{dL} = P^*, \qquad \frac{d^2E}{dL^2} = \frac{\varepsilon^2 W''(\varepsilon)}{L} \geq 0,$$

which shows that $P^* = 0$ is indeed the condition of extremality of $E(L)$, and that $L_{\rm opt}$ delivers the minimum of the energy. The fact that $P^* \neq 0$ the system can still lower its energy by either growth or resorption is illustrated in Fig. 2.

Example 2.6.

Here we show how to construct the simplest n dimensional configuratiuonally prestressed elastic body. For analytical transparency we limit our attention to spherically symmetric configurations and geometrically linear elasticity.

Specifically, the goal is to obtain a nontrivial radial stationary extremal on the ball $B(0,R) \subset \mathbb{R}^n$ of the functional

$$E[\boldsymbol{u}] = \int_{B(0,R)} W(\boldsymbol{x}, e(\boldsymbol{u}))d\boldsymbol{x}, \tag{2.30}$$

where the notation $e(\boldsymbol{u})$ was defined in (1.2), under the condition that the "body" $B(0,R)$ is not loaded by the "physical" forces in the sense that the extremal satisfies the boundary condition

$$\boldsymbol{P}\boldsymbol{n} = 0, \quad |\boldsymbol{x}| = R. \tag{2.31}$$

Note that (2.31) does not exclude the possibility that on the boundary we still have "configurational" loading in the sense that

$$\boldsymbol{P}^*\boldsymbol{n} \neq 0, \quad |\boldsymbol{x}| = R. \tag{2.32}$$

For our goal, it is sufficient to choose the simplest linearly elastic quadratic energy density of the form

$$W(\boldsymbol{x}, \boldsymbol{\varepsilon}) = \frac{1}{2}|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0(\boldsymbol{x})|^2, \tag{2.33}$$

where $\boldsymbol{\varepsilon}$ is a symmetric $n \times n$ matrix. The idea is to use the incompatible inhomogeneity $\boldsymbol{\varepsilon}_0(\boldsymbol{x})$ in the sense that

$$\mathrm{Ink}[\boldsymbol{\varepsilon}_0(\boldsymbol{x})] \neq 0 \tag{2.34}$$

where

$$\mathrm{Ink}[\boldsymbol{\varepsilon}(\boldsymbol{x})]_{ijkl} = \frac{\partial^2 \varepsilon_{ij}}{\partial x^k \partial x^l} - \frac{\partial^2 \varepsilon_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 \varepsilon_{lk}}{\partial x^i \partial x^j} - \frac{\partial^2 \varepsilon_{lj}}{\partial x^i \partial x^k}.$$

It is clear that (2.34) eliminates the trivial solution $\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{\varepsilon}_0$ with zero energy. We interpret the inhomogeneity $\boldsymbol{\varepsilon}_0(\boldsymbol{x})$ satisfying (2.34) as configurational prestress. To have a spherically symmetric example, we choose

$$\boldsymbol{\varepsilon}_0(\boldsymbol{x}) = a\widehat{\boldsymbol{x}} \otimes \widehat{\boldsymbol{x}}, \quad \widehat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}. \tag{2.35}$$

It is then straightforward to check that

$$\mathrm{Ink}[\boldsymbol{\varepsilon}_0]_{ijkl}(\boldsymbol{x}) = \frac{2a}{r^2}(\hat{x}^l\hat{x}^k\delta_{ij} - \hat{x}^i\hat{x}^k\delta_{lj} - \hat{x}^l\hat{x}^j\delta_{ik} + \hat{x}^i\hat{x}^j\delta_{lk} + \delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}) \neq 0, \tag{2.36}$$

except when $n = 2$, which case $\mathrm{Ink}[\boldsymbol{\varepsilon}_0]_{1122}(\boldsymbol{x}) = 2\pi\delta(\boldsymbol{x})$. This means that there is a local displacement field in $\mathbb{R}^2 \setminus \{0\}$, corresponding to the strain field (2.35), but no global one. This

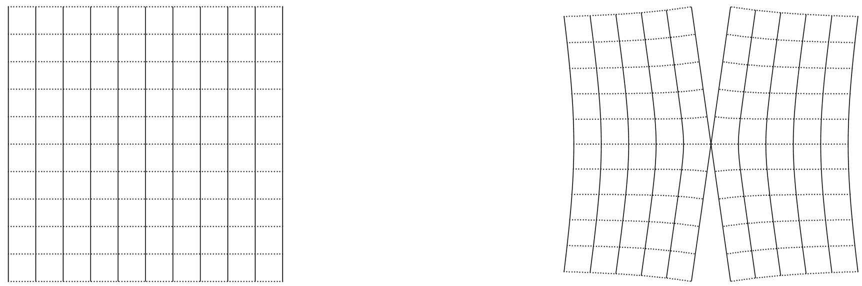

Figure 3: A perfect 2D grid represented on the left panel is transformed by the kinematically incompatible deformation strain field (2.35) into the discontinuous deformed grid shown on the right panel.

is illustrated in Fig. 3 by picturing the images of the Cartesian grid lines under the corresponding discontinous deformation $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x} + \epsilon \boldsymbol{u}(\boldsymbol{x})$, where

$$u_1(x_1, x_2) = x_1 + x_2 \arctan\left(\frac{x_2}{x_1}\right), \quad u_2(x_1, x_2) = x_2 - x_1 \arctan\left(\frac{x_2}{x_1}\right).$$

As the trivial candidate $e(\boldsymbol{u}) = \boldsymbol{\varepsilon}_0$ is not admissible, the minimizer of the energy (2.30) with free boundary conditions (2.31) will be a stationary extremal possessing positive energy as it will be unstressed "physically" but still prestressed "configurationally" in the sense that it will carry nonzero residual stresses.

The minimizer solves the following traction boundary value problem:

$$\begin{cases} \nabla(\nabla \cdot \boldsymbol{u}) + \Delta \boldsymbol{u} = 2\nabla \cdot \boldsymbol{\varepsilon}_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \boldsymbol{\sigma n} = (e(\boldsymbol{u}) - \boldsymbol{\varepsilon}_0)\boldsymbol{n} = 0, & \boldsymbol{x} \in \partial\Omega. \end{cases}$$

To obtain such a minimizer explicitly, when $\Omega = B(0, R)$—the ball in $\mathbb{R}^n$, centered at 0 with radius R, and $\boldsymbol{\varepsilon}_0(\boldsymbol{x})$ given by (2.35) we first rescale both $\boldsymbol{x}$ and $\boldsymbol{u}$ variables and set, without loss of generality, $a = 1$, $R = 1$. We will look for a radial solution

$$\boldsymbol{u}(\boldsymbol{x}) = \eta(r)\widehat{\boldsymbol{x}}. \tag{2.37}$$

Since $\nabla \boldsymbol{u}$ is symmetric, the function $\eta(r)$ should be chosen such that

$$\Delta \boldsymbol{u} = \nabla \cdot (\boldsymbol{x} \otimes \boldsymbol{x}), \tag{2.38}$$

or in terms of $\eta(r)$, such that

$$\eta''(r) + (n-1)\left(\frac{\eta(r)}{r}\right)' = \frac{(n-1)}{r}. \tag{2.39}$$

A solution of (2.39) that does not blow up at $r = 0$ must have the form

$$\eta(r) = br + \frac{n-1}{n} r \ln r. \tag{2.40}$$

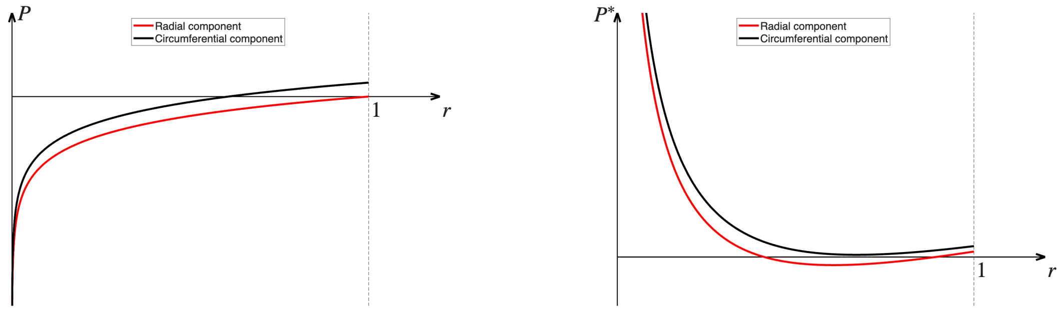

Figure 4: Radial and circumferential components of $\boldsymbol{P}$ (left panel) and $\boldsymbol{P}^*$ (right panel).

Finally, the zero traction boundary condition (2.31) gives $b = 1/n$, giving

$$\boldsymbol{P} = \left(1 + \frac{n-1}{n}\ln r\right)\boldsymbol{I}_n - \widehat{\boldsymbol{x}} \otimes \widehat{\boldsymbol{x}}, \tag{2.41}$$

$$\boldsymbol{P}^* = \frac{(n+2)(n-1)^2 \ln r - 1}{n^2}\widehat{\boldsymbol{x}} \otimes \widehat{\boldsymbol{x}} + \frac{(n+2)(n-1)^2(\ln r)^2 + 2(n^2-1)\ln r + n + 1}{2n^2}\boldsymbol{I}_n. \tag{2.42}$$

We can now compute the energy of our configurationally prestressed body by direct substitution, obtaining

$$E[\boldsymbol{u}] = \frac{|B(0,1)|}{2}\frac{n-1}{n^2} > 0, \tag{2.43}$$

where $|B(0,1)|$ denotes the n-dimensional volume of the unit ball in $\mathbb{R}^n$. As we have already mentioned, the positive value of the energy should be attributed to the nonzero external configurational loading

$$\boldsymbol{P}^*(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) = \frac{n-1}{2n^2}\widehat{\boldsymbol{x}}, \quad |\boldsymbol{x}| = R = 1.$$

When $n = 3$, we obtain that $\mathrm{Ink}[\boldsymbol{\varepsilon}_0]$ has only the radial component equal to $-2/r^2$. The plots of the radial and circumferential components of $\boldsymbol{P}$ and $\boldsymbol{P}^*$, given by (2.41) and (2.42), respectively, are shown in Fig. 4 The nonzero radial component of $\boldsymbol{P}^*$ on the boundary can be thought as a factor ensuring the embedding of the incompatibility $\mathrm{Ink}[\boldsymbol{\varepsilon}_0]$ into the body during the process of its imaginary growth through surface deposition.

Note that the configurational prestress can be attributed to the incompatibility of the embedded inhomogeneity of $\boldsymbol{\varepsilon}_0(\boldsymbol{x})$ in general. Indeed, if we recall the orhogonal decompostion of the Hilbert space $\mathscr{H} = L^2(\Omega; \mathrm{Sym}(\mathbb{R}^n))$

$$\mathscr{H} = \mathscr{E} \oplus \mathscr{J}, \quad \mathscr{E} = \{e(\boldsymbol{u}) : \boldsymbol{u} \in H^1(\Omega; \mathbb{R}^n)\}, \; \mathscr{J} = \{\boldsymbol{\sigma} \in \mathscr{H} : \nabla \cdot \boldsymbol{\sigma} = 0, \; \boldsymbol{\sigma}|_{\partial\Omega}\boldsymbol{n} = 0\}. \tag{2.44}$$

Therefore we can always write any field in $\mathscr{H}$ as a sum of a field in $\mathscr{E}$ and a field in $\mathscr{J}$. Specifically, for $\boldsymbol{\varepsilon}_0 \in \mathscr{H}$ we can find $\boldsymbol{u}_0 \in H^1(\Omega; \mathbb{R}^n)$ and $\boldsymbol{\sigma}_0 \in \mathscr{J}$, such that

$$\boldsymbol{\varepsilon}_0(\boldsymbol{x}) = e(\boldsymbol{u}_0) - \boldsymbol{\sigma}_0.$$

Then the energy minimizer will be $\boldsymbol{u} = \boldsymbol{u}_0$, the stress will be $\boldsymbol{\sigma}_0 = e(\boldsymbol{u}_0) - \boldsymbol{\varepsilon}_0(\boldsymbol{x})$ and the minimum value of the energy will be

$$E[\boldsymbol{u}_0] = \frac{1}{2}\|\boldsymbol{\sigma}_0\|^2_{L^2(\Omega)}.$$

The incompatible component $\boldsymbol{\sigma}_0$ can be found by solving the system

$$\begin{cases} \mathrm{Ink}[\boldsymbol{\sigma}_0] = -\mathrm{Ink}[\boldsymbol{\varepsilon}_0], & \boldsymbol{x} \in \Omega \\ \nabla \cdot \boldsymbol{\sigma}_0 = 0, & \boldsymbol{x} \in \Omega, \\ \boldsymbol{\sigma}_0 \boldsymbol{n} = 0, & \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{2.45}$$

The decomposition (2.44) implies that the boundary value problem (2.45) has a unique solution on simply connected domains, since $\mathrm{Ink}[\boldsymbol{\varepsilon}] = 0$ implies $\boldsymbol{\varepsilon} = e(\boldsymbol{v})$ for some $\boldsymbol{v} \in H^1$. Thus, $\boldsymbol{\sigma}_0$, and therefore the energy $E[\boldsymbol{u}_0]$ depends only on $\mathrm{Ink}[\boldsymbol{\varepsilon}_0]$. In fact,

$$\boldsymbol{P}^*\boldsymbol{n} = W\boldsymbol{n} = \frac{1}{2}|\boldsymbol{\sigma}_0|^2\boldsymbol{n}, \text{ on } \partial\Omega,$$

which allows us to conclude that our "physically unloaded" body was ultimately "loaded" by configurational forces.

2.3 Generalized Noether formula

We now summarize our general discussion by formulating the theorem:

THEOREM **2.7** (Generalized Noether formula)**.** *If $\boldsymbol{y}$ is a piecewise C^2 extremal in the sense of Definition 2.3, whose only singularity is a smooth surface Σ of jump discontinuity*[1] *of $\nabla\boldsymbol{y}(\boldsymbol{x})$, then the first variation δE of (2.1) under the action of a C^1 transformation (2.4) is given by*

$$\delta E = \int_{\partial\Omega} \{\boldsymbol{P}\boldsymbol{n} \cdot \delta\boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n} \cdot \delta\boldsymbol{x}\} dS - \int_{\Sigma} p^*_{\Sigma}\boldsymbol{n} \cdot \delta\boldsymbol{x}\, dS, \tag{2.46}$$

where p^*_{Σ} is given by (2.23). If the extremal $\boldsymbol{y}$ is stationary, then $p^*_{\Sigma} = 0$ in (2.46).

Remark 2.8. *One can also say that formula*

$$\delta E = \int_{\partial\Omega} \{\boldsymbol{P}\boldsymbol{n} \cdot \delta\boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n} \cdot \delta\boldsymbol{x}\} dS(\boldsymbol{x}), \tag{2.47}$$

holds for any Lipschitz stationary configuration. However, the right-hand side in (2.47) can no longer be understood as an integral of a functions over a surface, since $\boldsymbol{P}$ and $\boldsymbol{P}^*$ are not defined on $\partial\Omega$, when $\boldsymbol{y}$ is merely Lipschitz. However, if we additionally require that $\boldsymbol{y}$ be of class C^2 on a neighborhood of $\partial\Omega$, then the right-hand side in formula (2.47) is understood in the classical pointwise sense. We have used this remark in [38].

[1]The surface Σ does not need to be connected, and hence the case of several surfaces of jump discontinuity is included.

Theorem 2.7 reminds us that the classical condition of vanishing of the first variation is not satisfied in general for stationary configurations, but only for the variations that vanish on the boundary. It also points to the fact that in the case of extremals that are not stationary, as for instance occurs in dynamics, the vanishing of the first variation requires additionally that the defects (the discontinuities of $\nabla \boldsymbol{y}$) are not perturbed ($\delta \boldsymbol{x} = 0$ on Σ).

Finally, we mention that one can rewrite (2.47) in a form that makes it clear that only the part of the virtual graph-displacement $(\delta \boldsymbol{x}, \delta \boldsymbol{y})$ that moves the graph of $\boldsymbol{y}(\boldsymbol{x})$ as a geometric object contributes to the integral on the right-hand side of (2.47). Indeed, we observe that

$$T_{(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x}))}\partial\Gamma = \{(\boldsymbol{\tau}, \nabla \boldsymbol{y}(\boldsymbol{x})\boldsymbol{\tau}) : \boldsymbol{\tau} \in T_{\boldsymbol{x}}\partial\Omega\}$$

is the tangent space to the boundary of the graph of $\boldsymbol{y}(\boldsymbol{x})$ at the point $(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}))$, $\boldsymbol{x} \in \partial\Omega$. We then compute, using formula (2.8),

$$(\boldsymbol{P}^*\boldsymbol{n}, \boldsymbol{P}\boldsymbol{n}) \cdot (\boldsymbol{\tau}, \boldsymbol{F}\boldsymbol{\tau}) = \boldsymbol{P}^*\boldsymbol{n} \cdot \boldsymbol{\tau} + \boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{F}\boldsymbol{\tau} = W\boldsymbol{n} \cdot \boldsymbol{\tau} - \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{\tau} + \boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{F}\boldsymbol{\tau} = 0,$$

where we have used the shorthand $\boldsymbol{F}$ to denote $\nabla \boldsymbol{y}(\boldsymbol{x})$, and where $\boldsymbol{P}$ and $\boldsymbol{P}^*$ are evaluated at $(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x}))$. This shows that the vector $(\boldsymbol{P}^*\boldsymbol{n}, \boldsymbol{P}\boldsymbol{n}) \in \mathbb{R}^{m+n}$ is orthogonal to $\partial\Gamma$ at every point of $\partial\Gamma$. Therefore, we can write

$$\delta E = \int_{\partial\Omega} \{(\boldsymbol{P}^*\boldsymbol{n}, \boldsymbol{P}\boldsymbol{n}) \cdot \mathscr{P}_{(\partial\Gamma)^\perp}(\boldsymbol{x})(\delta \boldsymbol{x}, \delta \boldsymbol{y})\} dS, \tag{2.48}$$

where $\mathscr{P}_{(\partial\Gamma)^\perp}(\boldsymbol{x})$ denotes the orthogonal projection from $\mathbb{R}^{m+n}$ onto $(T_{(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x}))}\partial\Gamma)^\perp$.

2.4 Case study: dynamics

In elastodynamics the deformation is time dependent $\boldsymbol{y} = \boldsymbol{y}(t, \boldsymbol{x})$ and the functional $E[\boldsymbol{y}]$ is Lagrange's action integral

$$A[\boldsymbol{y}] = \int_0^T \left[\int_{\Omega(t)} \left\{ \frac{|\dot{\boldsymbol{y}}|^2}{2} - U(\nabla \boldsymbol{y}) \right\} d\boldsymbol{x} \right] dt, \tag{2.49}$$

where $\Omega(t)$—an arbitrary moving volume. In this section we assume that the deformation $\boldsymbol{y}$ satisfies all assumptions of Theorem 2.7.

Using our general framework we can rewrite (2.49) as a space-time integral

$$A[\boldsymbol{y}] = \int_{\Omega^{n+1}} \left\{ \frac{|\dot{\boldsymbol{y}}|^2}{2} - U(\nabla \boldsymbol{y}) \right\} d\boldsymbol{q}, \tag{2.50}$$

where $q^0 = t$, $q^\alpha = x^\alpha$, $\alpha = 1, \ldots, n$ and

$$\Omega^{n+1} = \{\boldsymbol{q} = (t, \boldsymbol{x}) : t \in [0, T],\ \boldsymbol{x} \in \Omega(t)\}.$$

The dynamic extremals are solutions of

$$\mathfrak{E}_L(\boldsymbol{q}) = -{}^{n+1}\nabla \cdot \boldsymbol{\mathscr{P}} = 0, \tag{2.51}$$

[111], where the dynamic Piola stress $\mathscr{P} \in \mathbb{R}^{m\times(n+1)}$ is given by

$$\mathscr{P} = [\boldsymbol{v}, -\boldsymbol{P}], \tag{2.52}$$

where $\boldsymbol{v} = \dot{\boldsymbol{y}}$, and, as before, $\boldsymbol{P} = U_{\boldsymbol{F}}(\nabla \boldsymbol{y})$. In (2.51) we also introduced the notation

$$^{n+1}\nabla = \left(\frac{\partial}{\partial t}, \nabla\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right).$$

Equation (2.51) can be of course rewritten as the classical Newton's equations of motion (balance of linear momentum)

$$\frac{\partial \boldsymbol{v}}{\partial t} = \nabla \cdot \boldsymbol{P}. \tag{2.53}$$

At those points $\boldsymbol{q} \in \mathbb{R}^{n+1}$, at which $\boldsymbol{y}$ is of class C^2, identity (2.13) applies and we also have

$$\mathfrak{E}_L^*(\boldsymbol{q}) = -^{n+1}\nabla \cdot \mathscr{P}^* = 0, \tag{2.54}$$

where the dynamic Eshelby tensor is

$$\mathscr{P}^* = \begin{bmatrix} -e & \boldsymbol{P}^T\boldsymbol{v} \\ -\boldsymbol{F}^T\boldsymbol{v} & \frac{1}{2}|\boldsymbol{v}|^2\boldsymbol{I}_n - \boldsymbol{P}^* \end{bmatrix}, \tag{2.55}$$

where

$$e = \frac{1}{2}|\dot{\boldsymbol{y}}|^2 + U(\nabla \boldsymbol{y})$$

is the total energy density and, again, as before, $\boldsymbol{P}^* = U(\nabla \boldsymbol{y})\boldsymbol{I}_n - \boldsymbol{F}^T\boldsymbol{P}$.

Finally, observe that in (2.54) we refer to the space-time Lagrangian

$$L(\mathscr{F}) = |\boldsymbol{v}|^2/2 - U(\boldsymbol{F}),$$

where the analog of deformation gradient is

$$\mathscr{F} = [\boldsymbol{v}, \boldsymbol{F}].$$

The system of $n+1$ equation (2.54) can be represented by a sub system of n equations

$$\frac{\partial(\boldsymbol{F}^T\boldsymbol{v})}{\partial t} - \frac{1}{2}\nabla(|\boldsymbol{v}|^2) + \nabla \cdot \boldsymbol{P}^* = 0, \tag{2.56}$$

which represent balance of linear momentum and can also be written as

$$\boldsymbol{F}^T\frac{\partial \boldsymbol{v}}{\partial t} + \nabla \cdot \boldsymbol{P}^* = 0, \tag{2.57}$$

plus an additional equation

$$-\frac{\partial e}{\partial t} + \nabla \cdot (\boldsymbol{P}^T\boldsymbol{v}) = 0, \tag{2.58}$$

representing balance of energy. Note that (2.56), as well as its analog (2.57), can be viewed as a configurational version of balance of linear momentum (2.53).

However, as is well-known, in dynamics the smoothness of $\boldsymbol{y}(\boldsymbol{q})$ can be violated even for smooth initial data due, for instance, to the formation of shock waves [22]. In this case equation (2.54) is no longer valid, and the function $\mathfrak{E}_L^*$ becomes a distribution supported on a singular set of ${}^{n+1}\nabla\boldsymbol{y}(\boldsymbol{q})$. When the singular set is a jump discontinuity surface Σ, we obtain [111], according to (2.22) that

$$\mathfrak{E}_L^*(\boldsymbol{q}) = -\mathscr{P}_\Sigma^*\boldsymbol{N}^{\rm sh}\delta_\Sigma(\boldsymbol{q}), \tag{2.59}$$

where

$$\mathscr{P}_\Sigma^* = [\![L]\!] - \langle\boldsymbol{\mathscr{P}}_\pm, [\![\boldsymbol{\mathscr{F}}]\!]\rangle \tag{2.60}$$

and $\boldsymbol{N}^{\rm sh}$ denotes unit normal to an n-dimensional surface $\Sigma \subset \mathbb{R}^{n+1}$. The choice of sign in (2.60) is immaterial in view of (2.20) that in our dynamics context takes the form

$$[\![\boldsymbol{\mathscr{P}}]\!]\boldsymbol{N}^{\rm sh} = 0. \tag{2.61}$$

Equations (2.61) are known as the Rankine-Hugoniot conditions.

It is instructive to rewrite (2.60) in terms of the potential U and block-components $\boldsymbol{v}$ and $\boldsymbol{F}$ of $\boldsymbol{\mathscr{F}}$. Suppose that the surface $\Sigma \subset \mathbb{R}^{n+1}$ represents a moving discontinuity surface $S(t) \subset \mathbb{R}^n$ so that

$$\Sigma = \{(t, S(t)) \subset \mathbb{R}^{n+1} : t \in [0, T]\}.$$

Let $V^{\rm sh}(t, \boldsymbol{x})$ denote the normal velocity of $S(t)$ and $\boldsymbol{n}(\boldsymbol{x}, t)$ the unit normal to $S(t)$. Then the vector

$$\widehat{\boldsymbol{N}}^{\rm sh}(\boldsymbol{q}) = [-V^{\rm sh}(t, \boldsymbol{x}), \boldsymbol{n}^{\rm sh}(t, \boldsymbol{x})],$$

is normal to Σ. Therefore, the Rankine-Hugoniot conditions (2.61) can be written as

$$[\![\boldsymbol{v}]\!]V^{\rm sh} + [\![\boldsymbol{P}]\!]\boldsymbol{n}^{\rm sh} = 0. \tag{2.62}$$

Observe that the Hadamard relations (2.21) in dynamical case take the form

$$[\![\boldsymbol{v}]\!] = -V^{\rm sh}\boldsymbol{a}, \quad [\![\boldsymbol{F}]\!] = \boldsymbol{a} \otimes \boldsymbol{n}^{\rm sh}, \tag{2.63}$$

where $\boldsymbol{a}(t, \boldsymbol{x}) : S(t) \to \mathbb{R}^m$ is a smooth vector field. Eliminating $\boldsymbol{a}$ from (2.63) we obtain

$$V^{\rm sh}[\![\boldsymbol{F}]\!] = -[\![\boldsymbol{v}]\!] \otimes \boldsymbol{n}^{\rm sh}. \tag{2.64}$$

Using (2.62) and (2.64) we can rewrite (2.60) as

$$\mathscr{P}_\Sigma^* = \langle\{\!\!\{\boldsymbol{P}\}\!\!\}, [\![\boldsymbol{F}]\!]\rangle - [\![U]\!], \tag{2.65}$$

where

$$\{\!\!\{\boldsymbol{P}\}\!\!\} = \frac{1}{2}(\boldsymbol{P}_+ + \boldsymbol{P}_-).$$

Indeed, expanding the right-hand side of (2.60) we have

$$\mathscr{P}_\Sigma^* = \{\!\!\{\boldsymbol{v}\}\!\!\} \cdot [\![\boldsymbol{v}]\!] - [\![U]\!] - \boldsymbol{v}_\pm \cdot [\![\boldsymbol{v}]\!] + \langle\boldsymbol{P}_\pm, [\![\boldsymbol{F}]\!]\rangle = \langle\boldsymbol{P}_\pm, [\![\boldsymbol{F}]\!]\rangle - [\![U]\!] \mp \frac{1}{2}|[\![\boldsymbol{v}]\!]|^2.$$

Taking a dot product of $(2.63)_1$ with $[\![\boldsymbol{v}]\!]$ we obtain

$$|[\![\boldsymbol{v}]\!]|^2 = -V^{\mathrm{sh}}[\![\boldsymbol{v}]\!] \cdot \boldsymbol{a}.$$

Now using (2.62) we obtain

$$|[\![\boldsymbol{v}]\!]|^2 = [\![\boldsymbol{P}]\!]\boldsymbol{n}^{\mathrm{sh}} \cdot \boldsymbol{a} = \langle [\![\boldsymbol{P}]\!], [\![\boldsymbol{F}]\!] \rangle,$$

where we have used $(2.63)_2$ in the last equality above. Now, observing that

$$\boldsymbol{P}_{\pm} \mp \frac{1}{2}[\![\boldsymbol{P}]\!] = \{\!\!\{ \boldsymbol{P} \}\!\!\},$$

we obtain (2.65). Comparing (2.65) and (2.23) we introduce a "spatial" version $p^*_{S(t)}$ of the space-time quantity $\mathscr{P}^*_\Sigma$:

$$p^*_{S(t)} = -\mathscr{P}^*_\Sigma = [\![U]\!] - \langle \{\!\!\{ \boldsymbol{P} \}\!\!\}, [\![\boldsymbol{F}]\!] \rangle. \tag{2.66}$$

Elastodynamics provides an interesting case of application of Theorem 2.7 because one can take advantage of the specific quadratic dependence of the Lagrangian on $\boldsymbol{v} = \dot{\boldsymbol{y}}$. Therefore, it is natural to consider the family of deformations

$$\widetilde{\boldsymbol{x}} = \boldsymbol{x}, \quad \widetilde{\boldsymbol{y}} = \boldsymbol{y}, \quad \widetilde{t} = e^{\epsilon} t. \tag{2.67}$$

So we have $\delta \boldsymbol{y} = 0$, $\delta \boldsymbol{q} = t\boldsymbol{e}_0$ and the right-hand side of (2.46) becomes

$$\int_{\partial\Omega^{n+1}} t[-e, \boldsymbol{P}^T\boldsymbol{v}] \cdot \boldsymbol{N} d\mathcal{S} - \int_\Sigma t\mathscr{P}^*_\Sigma \boldsymbol{N}^{\mathrm{sh}} \cdot \boldsymbol{e}_0 d\mathcal{S}$$

where we used the fact that $(\boldsymbol{\mathscr{P}}^*)^T\boldsymbol{e}_0 = [-e, \boldsymbol{P}^T\boldsymbol{v}]$. We also observe that on $\partial\Omega^{n+1}$ we have

$$\boldsymbol{N} d\mathcal{S} = [-V_n, \boldsymbol{n}] dS dt, \tag{2.68}$$

where $\boldsymbol{N}(\boldsymbol{q})$ is the outward unit normal to $\partial\Omega^{n+1}$, $\boldsymbol{n}(t, \boldsymbol{x})$ is the outward unit normal to $\partial\Omega(t)$, and $V_n(t, \boldsymbol{x})$ is the normal velocity of $\partial\Omega(t)$. Similarly,

$$\boldsymbol{N}^{\mathrm{sh}} d\mathcal{S} = [-V^{\mathrm{sh}}, \boldsymbol{n}^{\mathrm{sh}}] dS dt,$$

Hence, the right-hand side of (2.46) can be written as

$$\int_0^T \int_{\partial\Omega(t)} t(V_n e + \boldsymbol{P}^T\boldsymbol{v} \cdot \boldsymbol{n}) dS dt - t \int_{\Omega(t)} e d\boldsymbol{x} \bigg|_0^T + \int_0^T \int_{S(t)} t V^{\mathrm{sh}} \mathscr{P}^*_\Sigma dS(t) dt.$$

Let us now compute the left-hand side in (2.46). Given that

$$\widetilde{\boldsymbol{y}}_\epsilon(\widetilde{t}, \boldsymbol{x}) = \boldsymbol{y}(e^{-\epsilon}\widetilde{t}, \boldsymbol{x}),$$

we can write

$$A[\widetilde{\boldsymbol{y}}_\epsilon] = \int_0^{e^\epsilon T} \int_{\Omega(t)} \left\{ \frac{e^{-2\epsilon}}{2} |\dot{\boldsymbol{y}}(e^{-\epsilon}\widetilde{t}, \boldsymbol{x})|^2 - U(\nabla \boldsymbol{y}) \right\} d\boldsymbol{x} d\widetilde{t}.$$

Changing variables $\widetilde{t} = e^{\epsilon} t$ we obtain

$$A[\widetilde{\boldsymbol{y}}_\epsilon] = \int_0^T \int_{\Omega(t)} \left\{ \frac{e^{-\epsilon}}{2} |\dot{\boldsymbol{y}}(t, \boldsymbol{x})|^2 - e^{\epsilon} U(\nabla \boldsymbol{y}) \right\} d\boldsymbol{x} dt.$$

Hence, the left-hand side in (2.46) is

$$\delta A = \left. \frac{dA[\widetilde{\boldsymbol{y}}_\epsilon]}{d\epsilon} \right|_{\epsilon=0} = - \int_0^T \int_{\Omega(t)} e d\boldsymbol{x} dt.$$

We conclude that for the family of deformations (2.67) the equation (2.46) can be rewritten in the form

$$\int_0^T \int_{\Omega(t)} e d\boldsymbol{x} dt = t \int_{\Omega(t)} e d\boldsymbol{x} \Big|_0^T - \int_0^T \int_{\partial\Omega(t)} t \{ e V_n + \boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{v} \} dS dt - \int_0^T \int_{S(t)} V^{\rm sh} t \mathscr{P}_\Sigma^* dS dt.$$

After differentiation in T and other simplifications we obtain

$$\frac{d}{dt} \int_{\Omega(t)} e d\boldsymbol{x} = \int_{\partial\Omega(t)} \boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{v} \, dS + \int_{\partial\Omega(t)} e V_n dS + \int_{S(t)} V^{\rm sh} \mathscr{P}_\Sigma^* dS. \tag{2.69}$$

This is nothing else but the energy balance relation, showing that the change of the total energy contained in a moving volume takes place because of the work performed by external surface tractions, the energy brought inside the body due to addition/removal of mass through the external boundary, and finally the energy produced or absorbed due to the mass exchange on the moving strain discontinuity.

Note that, due to thermodynamic reasons, on physically admissible discontinuities in passive media the energy can only dissipate and therefore we must necessarily have

$$V^{\rm sh} \mathscr{P}_\Sigma^* \le 0. \tag{2.70}$$

The meaning of the inequality (2.70) is that extremals in the case of dynamics cannot be expected to be stationary and the the condition ${}^{n+1}\nabla \cdot \boldsymbol{\mathscr{P}}^* = 0$ does not have to hold. More specifically, it means that ${}^{n+1}\nabla \cdot \boldsymbol{\mathscr{P}}^*$ can be represented by a distribution supported on a singular surface of the solution and subjected only to an inequality constraint (2.70). The conventional form of this constraint known from the theory of shock waves [111] can be obtained if we recall that by convention the direction of $\boldsymbol{n}^{\rm sh}$ is always chosen to point in the direction of the advance of the discontinuity, $V^{\rm sh} \ge 0$. Therefore, on a physically admissible strain discontinuity we must always have $\mathscr{P}_\Sigma^* \le 0$, or

$$p_{S(t)}^* = [\![U]\!] - \langle \{\!\!\{ \boldsymbol{P} \}\!\!\}, [\![\boldsymbol{F}]\!] \rangle \ge 0, \tag{2.71}$$

where $p_{S(t)}^*$ is defined in (2.66), and the "$\pm$" convention in (2.71) is such that the "$-$" side is behind the shock wave and the "$+$" side is in front.

3 Partial invariance

Throughout this section we assume that both W and the extremal are of class C^2. The main significance of the developed formalism lies in its usefulness for converting special scaling and invariance properties of variational problems into useful *formulas* satisfied by stationary configurations or almost regular extremals (as in Theorem 2.7(b)). The idea is to use particular deformations $\boldsymbol{\Phi}_\epsilon$ of the $(\boldsymbol{x}, \boldsymbol{y})$ space under which, due to the special properties of the Lagrangian $W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})$, the functional $E[T\Gamma]$ in (2.3) transforms *in a known way*.

In this section we collected several examples illustrating the idea that if $E[T\Gamma_\epsilon]$ can be simplified or tranformed by means of specific properties of the Lagrangian, then δE, computed as (2.2) will express this "partial symmetry", while equations (2.47) or (2.46) will represent the corresponding "partial conservation laws". In all these cases we show how such partial invariance of a problem can be used to obtain useful information about the solutions.

3.1 Parametric problems

An important set of examples concerns parametric variational integrals

$$E[\Sigma] = \int_D W(\boldsymbol{z}(\boldsymbol{t}), \nabla \boldsymbol{z}(\boldsymbol{t})) d\boldsymbol{t}. \tag{3.1}$$

The goal is to find an optimal n-dimensional surface $\Sigma = \{\boldsymbol{z}(\boldsymbol{t}) : \boldsymbol{t} \in D\} \subset \mathbb{R}^m$, where $D \subset \mathbb{R}^n$ is open, bounded and smooth. The solution is not supposed to depend on a particular choice of parametrization.

Definition 3.1. *Suppose that $\boldsymbol{t}(\boldsymbol{s})$ is any diffeomorphism between the closures of the open, bounded and smooth domains $\widetilde{D}$ and D and $\widetilde{\boldsymbol{z}}(\boldsymbol{s}) = \boldsymbol{z}(\boldsymbol{t}(\boldsymbol{s}))$. The Lagrangian W is called* parametric *if*

$$\int_D W(\boldsymbol{z}(\boldsymbol{t}), \nabla \boldsymbol{z}(\boldsymbol{t})) d\boldsymbol{t} = \int_{\widetilde{D}} W(\widetilde{\boldsymbol{z}}(\boldsymbol{s}), \nabla \widetilde{\boldsymbol{z}}(\boldsymbol{s})) d\boldsymbol{s}. \tag{3.2}$$

for any smooth function $\boldsymbol{z}(\boldsymbol{t})$.

The notion of the parametric Lagrangian can be reformulated as a a variational symmetry. Let $\boldsymbol{\theta} : D \to \mathbb{R}^n$, be an arbitrary smooth function, and let $\boldsymbol{s} = \boldsymbol{t} + \epsilon \boldsymbol{\theta}(\boldsymbol{t})$. Then $\widetilde{\boldsymbol{z}}(\boldsymbol{s})$ is a reparametrization of Σ for all sufficiently small in absolute value ϵ, if

$$\boldsymbol{z}(\boldsymbol{t}) = \widetilde{\boldsymbol{z}}(\boldsymbol{s}) = \widetilde{\boldsymbol{z}}(\boldsymbol{t} + \epsilon \boldsymbol{\theta}(\boldsymbol{t})).$$

By definition of the parametric variartional integral equation (3.2) must hold. But then, formula (2.9) applied to the transformation

$$\boldsymbol{\Phi}_\epsilon(\boldsymbol{t}, \boldsymbol{z}) = (\boldsymbol{t} + \epsilon \boldsymbol{\theta}(\boldsymbol{t}), \boldsymbol{z}), \quad \delta \boldsymbol{t} = \boldsymbol{\theta}, \quad \delta \boldsymbol{z} = 0,$$

yields taking into account that W is independent of $\boldsymbol{t}$,

$$\int_D \langle \boldsymbol{P}^*, \nabla \boldsymbol{\theta} \rangle d\boldsymbol{t} = 0.$$

Since, both the domain D and the vector field $\boldsymbol{\theta}$ were arbitrary, we conclude that

$$\boldsymbol{P}^*(\boldsymbol{z}, \boldsymbol{F}) = 0 \tag{3.3}$$

identically. Equation (3.3) is then a universal relation to be satisfied for any solution of a parametric problem.

It is now easy to see what it takes for the problem to be parametric. If we fix $\boldsymbol{z} \in \mathbb{R}^m$ and $\boldsymbol{F} \in \mathbb{R}^{m\times n}$, and differentiate the function

$$GL(n) \ni \boldsymbol{A} \mapsto \Phi(\boldsymbol{A}) = \frac{W(\boldsymbol{z}, \boldsymbol{F}\boldsymbol{A})}{\det \boldsymbol{A}}$$

we obtain

$$\nabla\Phi(\boldsymbol{A}) = \frac{\boldsymbol{F}^T W_{\boldsymbol{F}}(\boldsymbol{z}, \boldsymbol{F}\boldsymbol{A}) - W(\boldsymbol{z}, \boldsymbol{F}\boldsymbol{A})\boldsymbol{A}^{-T}}{\det \boldsymbol{A}} = \frac{\boldsymbol{A}^{-T}}{\det \boldsymbol{A}} \boldsymbol{P}^*(\boldsymbol{z}, \boldsymbol{F}\boldsymbol{A}) = 0.$$

We can then conclude that if $W(\boldsymbol{z}, \boldsymbol{F})$ is a parametric Lagrangian, then it must have the property

$$W(\boldsymbol{z}, \boldsymbol{F}\boldsymbol{A}) = W(\boldsymbol{z}, \boldsymbol{F}) \det \boldsymbol{A} \tag{3.4}$$

for any $\boldsymbol{A} \in GL^+(n)$. Conversely, if (3.4) holds for all $\boldsymbol{A} \in GL^+(n)$, then, differentiating (3.4) in $\boldsymbol{A}$ at $\boldsymbol{A} = \boldsymbol{I}_n$, we obtain $\boldsymbol{P}^*(\boldsymbol{z}, \boldsymbol{F}) = 0$. Therefore, the algebraic property (3.4) is equivalent to (3.3) and gives a complete characterization of parametric Lagrangians. Note that a well known example of parametric invariance in the context of nonlinear elasticity is the invariance with respect to "material remodeling", see for instance [26].

To show that parametric problems are ubiquitous, consider again the transformations $\boldsymbol{\Phi}_\epsilon(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}, \epsilon), \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{y}, \epsilon))$ from (2.4) which treat the $(\boldsymbol{x}, \boldsymbol{y})$-space $\mathfrak{X}$ as a single entity. We recall that from this standpoint the field $\boldsymbol{y}(\boldsymbol{x})$ is determined by its graph $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$. The crucial observation is that the *non-parametric* variational integral (2.1) can be written as a *parametric* integral by making an arbitrary change of variables

$$\boldsymbol{x} = \boldsymbol{\eta}(\boldsymbol{t}), \tag{3.5}$$

where $\boldsymbol{\eta} : \overline{D} \to \overline{\Omega}$ is a diffeomorphism, and D is an arbitrary domain, such that $\overline{D}$ is diffeomorphic to $\overline{\Omega}$.

Indeed, since all transformations (3.5) do not change the graph of $\boldsymbol{y}$, i.e.

$$\Gamma = \{(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})) : \boldsymbol{x} \in \Omega\} = \{(\boldsymbol{\eta}(\boldsymbol{t}), \boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t}))) : \boldsymbol{t} \in D\}, \tag{3.6}$$

it is natural to introduce the notation

$$\boldsymbol{z}(\boldsymbol{t}) = [\boldsymbol{\eta}(\boldsymbol{t}); \boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t}))],$$

where $[a; b]$ denotes the column-vector $\begin{bmatrix} a \\ b \end{bmatrix}$. We can now write

$$\nabla\boldsymbol{z}(\boldsymbol{t}) = [\nabla\boldsymbol{\eta}; \nabla\boldsymbol{y}\nabla\boldsymbol{\eta}] = [(\nabla\boldsymbol{z})_1; (\nabla\boldsymbol{z})_2],$$

and therefore,

$$E[\boldsymbol{y}] = \int_D W(\boldsymbol{z}(\boldsymbol{t}), (\nabla\boldsymbol{z})_2(\nabla\boldsymbol{z})_1^{-1}) \det((\nabla\boldsymbol{z})_1) d\boldsymbol{t} =: \mathscr{E}[\boldsymbol{z}; D].$$

Thus, the original non-parametric Lagrangian $W(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F})$ has been replaced by the parametric Lagrangian operating in the extended space

$$\mathscr{W}(\boldsymbol{z},\boldsymbol{\mathscr{F}})=W(\boldsymbol{z},\boldsymbol{F}_2\boldsymbol{F}_1^{-1})\det\boldsymbol{F}_1,$$

where $\boldsymbol{z}\in\mathbb{R}^{n+m}$, $\boldsymbol{\mathscr{F}}=[\boldsymbol{F}_1;\boldsymbol{F}_2]\in\mathbb{R}^{(m+n)\times n}$, and $\boldsymbol{F}_1$ is the upper $n\times n$ submatrix of $\boldsymbol{\mathscr{F}}$. Note that even if the original Lagrangian was smooth on $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}$, the new Lagrangian is smooth only on a subset $\mathscr{O}$ of $\mathbb{R}^{n+m}\times\mathbb{R}^{(m+n)\times n}$, on which $\boldsymbol{F}_1$ is invertible.

By direct differentiation we obtain

$$\boldsymbol{\mathscr{P}}(\boldsymbol{z},\boldsymbol{\mathscr{F}})=\mathscr{W}_{\boldsymbol{\mathscr{F}}}=\begin{bmatrix}\boldsymbol{P}^*(\boldsymbol{z},\boldsymbol{F}_2\boldsymbol{F}_1^{-1})\mathrm{cof}\boldsymbol{F}_1\\ \boldsymbol{P}(\boldsymbol{z},\boldsymbol{F}_2\boldsymbol{F}_1^{-1})\mathrm{cof}\boldsymbol{F}_1\end{bmatrix}. \tag{3.7}$$

Note that such a higher dimensional Piola stress $\boldsymbol{\mathscr{P}}$ mixes the classical low dimensional Piola and Eshelby stresses. We also have, according to (3.3) , that

$$\boldsymbol{\mathscr{P}}^*(\boldsymbol{z},\boldsymbol{\mathscr{F}})=0$$

where

$$\boldsymbol{\mathscr{P}}^*(\boldsymbol{z},\boldsymbol{\mathscr{F}})=\mathscr{W}(\boldsymbol{z},\boldsymbol{\mathscr{F}})\boldsymbol{I}_{n+m}-\boldsymbol{\mathscr{F}}^T\boldsymbol{\mathscr{P}}(\boldsymbol{z},\boldsymbol{\mathscr{F}}).$$

We also see that in view of the parametric nature of the problem

$$\mathscr{W}(\boldsymbol{z},\boldsymbol{\mathscr{F}}\boldsymbol{Q})=\mathscr{W}(\boldsymbol{z},\boldsymbol{\mathscr{F}})\det\boldsymbol{Q} \tag{3.8}$$

for any invertible $n\times n$ matrix $\boldsymbol{Q}$.

Finally, we show that for every stationary configuration $\boldsymbol{y}(\boldsymbol{x})$ of the original functional (2.1) and any smooth diffeomorphism $\boldsymbol{\eta}:D\to\Omega$ the function $\boldsymbol{z}(\boldsymbol{t})=(\boldsymbol{\eta}(\boldsymbol{t}),\boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t})))$ is an equilibrium configuration of $\mathscr{E}[\boldsymbol{z};D]$. Intuitively, the statement is apparent, since any weak outer variation of $\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y})$ is a combination of an inner and outer variations for $\boldsymbol{y}(\boldsymbol{x})$ and it can also be verified by a direct calculation. Indeed, the extended Euler-Lagrange equation

$$\nabla_{\boldsymbol{t}}\cdot\mathscr{W}_{\boldsymbol{\mathscr{F}}}(\boldsymbol{z},\boldsymbol{\mathscr{F}})=\mathscr{W}_{\boldsymbol{z}}$$

can be written in terms of $\boldsymbol{y}(\boldsymbol{x})$ as a system

$$\begin{cases}\nabla_{\boldsymbol{t}}\cdot[\boldsymbol{P}^*(\boldsymbol{\eta}(\boldsymbol{t}),\boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t})),\nabla_{\boldsymbol{x}}\boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t})))\mathrm{cof}\nabla_{\boldsymbol{t}}\boldsymbol{\eta}(\boldsymbol{t})]=W_{\boldsymbol{x}}\det\nabla_{\boldsymbol{t}}\boldsymbol{\eta}(\boldsymbol{t}),\\ \nabla_{\boldsymbol{t}}\cdot(\boldsymbol{P}(\boldsymbol{\eta}(\boldsymbol{t}),\boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t})),\nabla_{\boldsymbol{x}}\boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t})))\mathrm{cof}\nabla_{\boldsymbol{t}}\boldsymbol{\eta}(\boldsymbol{t}))=W_{\boldsymbol{y}}\det\nabla_{\boldsymbol{t}}\boldsymbol{\eta}(\boldsymbol{t}).\end{cases} \tag{3.9}$$

It is then clear that a smooth change of variables $\boldsymbol{x}=\boldsymbol{\eta}(\boldsymbol{t})$ converts the system (3.9) into an equivalent already familiar system

$$\mathfrak{E}^*(W)=0,\quad\mathfrak{E}(W)=0.$$

3.2 Invariant integrals

Suppose that the energy density W is homogeneous, i.e., does not depend explicitly on $\boldsymbol{x}$. Therefore, we apply the transformation (2.4), given by

$$\boldsymbol{\Phi}_\epsilon(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{x}+\epsilon\boldsymbol{a},\boldsymbol{y}). \tag{3.10}$$

Then according to (2.13), we can write

$$-\nabla\cdot\boldsymbol{P}^* = \mathfrak{E}^*_W(\boldsymbol{x}) = -\nabla\boldsymbol{y}^T\mathfrak{E}_W(\boldsymbol{x}) = 0,$$

and for any piecewise smooth subdomain $V\subset\Omega$, we must have

$$\int_{\partial V}\boldsymbol{P}^*\boldsymbol{n}\,dS = 0. \tag{3.11}$$

To show that such statement can be highly nontrivial, it sufficient to refer to the J-integral in fracture mechancis which in two space dimensions $\boldsymbol{n}dl = (dx_2,-dx_1)$, is defined as a vector with two components

$$J_1 = \int_C\left\{W dx_2 - \boldsymbol{t}\cdot\frac{\partial\boldsymbol{y}}{\partial x_1}dl\right\},\qquad J_2 = -\int_C\left\{W dx_1 + \boldsymbol{t}\cdot\frac{\partial\boldsymbol{y}}{\partial x_2}dl\right\}.$$

Here we use the notation $\boldsymbol{t}=\boldsymbol{P}\boldsymbol{n}$ for traction forces and C is any path in Ω. It is then clear that the general definition of J-integral is

$$\boldsymbol{J} = \int_{\partial U}\boldsymbol{P}^*\boldsymbol{n}\,dS, \tag{3.12}$$

where the subdomain $U\subset\Omega_0$. Due to (3.11), the value of the integral $\boldsymbol{J}$ does not change if we deform the set U smoothly. If the set U contains a connected component of the singular set S in its interior, the value of $\boldsymbol{J}$ does not have to vanish and it is invariant as long as the boundary of the deformed set does not intersects the singular set S. More formally, one can say that the J-integral (3.12) is either loop homotopy invariant or fixed end-point homotopy invariant in $\Omega = \Omega_0\setminus S$. The most important example of a singular set S of this type is provided by the mechanical fracture or crack set $S\subset\Omega_0$, usually represented by one or more smooth surfaces of displacement discontinuity with elastic deformation $\boldsymbol{y}$ remaining smooth (at least C^2) on $\Omega=\overline{\Omega}_0\setminus S$. Note that the homotopy invariance of the J-integral is shown here under more general assumptions than it is usually done in fracture mechanics [92, 12, 16, 24, 18] in that we permit the energy density to depend on both $\boldsymbol{y}$ and $\boldsymbol{F}$.

Another example of this type comes naturally if we replace translational invariance by rotational invariance. In mechanics the corresponding homotopy-invariant quantity, called the L-integral in [12], was introduced in the context of geometrically linear elasticity that does not make a distinction between a reference and a deformed configuration. To give a variational interpretation of the now classical L-integral, we will have to define isotropy in a way that is not quite natural from the point of view of finite elasticity.

Definition 3.2. *We say that the energy density $W(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F})$ is isotropic if*

$$W(\boldsymbol{R}\boldsymbol{x},\boldsymbol{R}\boldsymbol{y},\boldsymbol{R}\boldsymbol{F}\boldsymbol{R}^T) = W(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F}),\quad \forall\boldsymbol{R}\in SO(3). \tag{3.13}$$

It is easy to check that the transformation group (2.4), given by

$$\boldsymbol{\Phi}_\epsilon(\boldsymbol{x},\boldsymbol{y})=(e^{\epsilon\star\boldsymbol{\omega}}\boldsymbol{x},e^{\epsilon\star\boldsymbol{\omega}}\boldsymbol{y}),$$

is a variational symmetry. Here the 3×3 antisymmetric matrix $\star\boldsymbol{\omega}$ is a special case of the Hodge-star operator[2] defined in our specific case of $\mathbb{R}^3$ by $(\star\boldsymbol{\omega})\boldsymbol{v}=\boldsymbol{v}\times\boldsymbol{\omega}$. It is clear that the left-hand side in (2.47) is zero and that $\delta\boldsymbol{x}=\boldsymbol{x}\times\boldsymbol{\omega}$ and $\delta\boldsymbol{y}=\boldsymbol{y}\times\boldsymbol{\omega}$. Taking into account that $\boldsymbol{\omega}\in\mathbb{R}^3$ is arbitrary, we obtain

$$\int_{\partial V}\{\boldsymbol{P}\boldsymbol{n}\times\boldsymbol{y}+\boldsymbol{P}^*\boldsymbol{n}\times\boldsymbol{x}\}dS=0, \tag{3.14}$$

where $V\subset\Omega$ is any piecewise smooth subdomain. Equation (3.14) suggests that

$$\boldsymbol{L}=\int_{\Sigma}\{\boldsymbol{P}\boldsymbol{n}\times\boldsymbol{y}+\boldsymbol{P}^*\boldsymbol{n}\times\boldsymbol{x}\}dS,$$

where Σ is any closed surface in Ω is a homotopy-invariant integral. The nonzero values of $\boldsymbol{L}$ come from noncontractible closed surfaces that are usually the boundaries of smooth domains in Ω_0 containg one or more components of the crack set S. We point out again that the homotopy invariance of the L-integral is proved here under more general assumptions than in [12], in that we permit the energy density to depend (smoothly) on $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{F}$, as long as the isotropy property (3.13) holds.

Yet another example of an invariant integral in linear elasticity emerges if we note that the corresponding energy density is a quadratic function of the displacement gradient. The corresponding scaling invariance property can be also formulated as a classical variational symmetry. Suppose for simplicity that $W=W(\boldsymbol{F})$ is independent of either $\boldsymbol{x}$ or $\boldsymbol{y}$. We will also assume, more generally, that

$$W(\lambda\boldsymbol{F})=\lambda^pW(\boldsymbol{F}),\quad \forall\lambda>0,\ \forall\boldsymbol{F}\in\mathbb{R}^{m\times n}, \tag{3.15}$$

for some $p\neq 0$. It is easy to find a scaling law that $W(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F})$ would have to satisfy to possess a variational symmetry:

$$W(\lambda^{-\frac{n}{p}}\boldsymbol{x},\lambda^{1-\frac{n}{p}}\boldsymbol{y},\lambda\boldsymbol{F})=\lambda^pW(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F}).$$

It is straightforward to verify that the corresponding transformation group

$$\boldsymbol{\Phi}_\epsilon(\boldsymbol{x},\boldsymbol{y})=(e^{\epsilon}\boldsymbol{x},e^{\frac{p-n}{p}\epsilon}\boldsymbol{y})$$

is a variational symmetry. Then applying (2.47) with $\delta\boldsymbol{x}=\boldsymbol{x}$, $\delta\boldsymbol{y}=\frac{p-n}{p}\boldsymbol{y}$ we obtain that

$$\int_{\partial V}\left\{\boldsymbol{P}^*\boldsymbol{n}\cdot\boldsymbol{x}+\frac{p-n}{p}\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{y}\right\}dS=0. \tag{3.16}$$

[2]The Hodge-star operator maps p-forms ω on an n-dimensional Euclidean vector space V into twisted $n-p$ forms $\star\omega$, which requires a choice of an orientation on V. It is defined by to the rule $\omega[v_1,\ldots,v_p]=\star\omega[v_{p+1},\ldots,v_n]$ for any positively oriented orthonormal basis $[v_1,\ldots,v_n]$ of V. Here we use the convention $\omega[v_1,\ldots,v_p]=\omega_{i_1\ldots i_p}v_1^{i_1}\cdots v_p^{i_p}$, where the summation over repeated indices is assumed. The object $\star\omega$ is a twisted $n-p$ form because it changes sign, when the orientation of V changes, while ordinary forms do not, see [9, Section 4.5].

In the special case $p = 2$, $n = 2$, (3.16) reduces to the classical M-integral which we write in the form

$$M_{2D} = \int_{\Sigma} \boldsymbol{P}^* \boldsymbol{n} \cdot \boldsymbol{x} \, dS,$$

If $p = 2$ and $n = 3$, we obtain a more complex expression

$$M_{3D} = \int_{\Sigma} \left\{ \boldsymbol{P}^* \boldsymbol{n} \cdot \boldsymbol{x} - \frac{1}{2} \boldsymbol{P} \boldsymbol{n} \cdot \boldsymbol{y} \right\} dS,$$

which can be viewed as a generalization of the relation first obtained in [12]. For more information about path independent integrals in elastostatics and elastodynamics and their relation to Noether theorems, see [30, 46, 24, 45, 70, 69, 81].

3.3 Case study: Pohozaev-type identities

In this section we show how to use Noether's formula (2.47) to generalize the so called Pohozaev's identity which is often used in the proof of various uniqueness results for nonlinear PDEs [86, 113]. Consider a boundary value problem for a nonlinear PDE of the form

$$\begin{cases} \nabla \cdot W_{\boldsymbol{F}}(\nabla \boldsymbol{y}(\boldsymbol{x})) + \Phi_{\boldsymbol{y}}(\boldsymbol{y}(\boldsymbol{x})) = 0, & \boldsymbol{x} \in \Omega, \\ \boldsymbol{y}(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \partial\Omega, \end{cases} \tag{3.17}$$

where the potentials Φ and W are of class C^1, $\Phi(0) = 0$, $W(\boldsymbol{F})$ is positively homogeneous of degree $p \geq 1$, and $W(\boldsymbol{u} \otimes \boldsymbol{v}) \geq 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{R}^m$. We also assume that $\Omega \subset \mathbb{R}^n$ is star-shaped with respect to the origin[3]. The key observation is that any solution $\boldsymbol{y} \in C^2(\overline{\Omega}; \mathbb{R}^m)$ must be a stationary extremal of the functional

$$E[\boldsymbol{y}] = \int_{\Omega} \{W(\nabla \boldsymbol{y}) - \Phi(\boldsymbol{y})\} d\boldsymbol{x}. \tag{3.18}$$

To obtain the generalized Pohozaev's identities we apply Noether's formula (2.47) to the functional (3.18) for the family of transformations

$$\widetilde{\boldsymbol{x}} = e^{\epsilon} \boldsymbol{x}, \quad \widetilde{\boldsymbol{y}} = e^{A\epsilon} \boldsymbol{y}, \tag{3.19}$$

where $A \in \mathbb{R}$ is arbitrary. In this case, $\delta \boldsymbol{x} = \boldsymbol{x}$, $\delta \boldsymbol{y} = A \boldsymbol{y}$, and the Noether's formula (2.47) becomes

$$\delta E = \int_{\partial\Omega} \boldsymbol{P}^* \boldsymbol{n} \cdot \boldsymbol{x} \, dS, \tag{3.20}$$

since $\delta \boldsymbol{y}(\boldsymbol{x}) = A \boldsymbol{y}(\boldsymbol{x}) = 0$ on $\partial\Omega$, in view of the boundary conditions. Here

$$\boldsymbol{P}^*(\boldsymbol{x}) = W(\nabla \boldsymbol{y}) \boldsymbol{I}_n - (\nabla \boldsymbol{y})^T W_{\boldsymbol{F}}(\nabla \boldsymbol{y}), \quad \boldsymbol{x} \in \partial\Omega,$$

since $\Phi(0) = 0$. We also compute

$$\delta E = \int_{\Omega} \{n(W(\nabla \boldsymbol{y}) - \Phi(\boldsymbol{y})) + (A - 1)\langle \nabla \boldsymbol{y}, W_{\boldsymbol{F}}(\nabla \boldsymbol{y}) \rangle - A \Phi_{\boldsymbol{y}}(\boldsymbol{y}) \cdot \boldsymbol{y}\} d\boldsymbol{x}.$$

[3] We are free to choose the origin arbitrarily because the PDE in (3.17) does not depend on $\boldsymbol{x}$ explicitly.

Since, A is arbitrary, Noether's formula (3.20) implies the two generalized Pohozaev's identities for C^2 solutions of the boundary value problem (3.17):

$$\int_{\Omega}\{\langle\nabla\boldsymbol{y}, W_{\boldsymbol{F}}(\nabla\boldsymbol{y})\rangle-\Phi_{\boldsymbol{y}}(\boldsymbol{y})\cdot\boldsymbol{y}\}d\boldsymbol{x}=0 \tag{3.21}$$

and

$$\int_{\Omega}\{n(W(\nabla\boldsymbol{y})-\Phi(\boldsymbol{y}))-\langle W_{\boldsymbol{F}}(\nabla\boldsymbol{y}),\nabla\boldsymbol{y}\rangle\}d\boldsymbol{x}= \int_{\partial\Omega}\{(W(\boldsymbol{a}\otimes\boldsymbol{n})-\langle W_{\boldsymbol{F}}(\boldsymbol{a}\otimes\boldsymbol{n}),\boldsymbol{a}\otimes\boldsymbol{n}\rangle)(\boldsymbol{n}\cdot\boldsymbol{x})\}dS, \tag{3.22}$$

where we have used the boundary conditions in (3.17), and denoted $\boldsymbol{a}(\boldsymbol{x})=\partial\boldsymbol{y}/\partial\boldsymbol{n}\in C^1(\partial\Omega;\mathbb{R}^m)$. The positive p-homogeneity of $W(\boldsymbol{F})$ allows us to simplify Pohozaev's identities:

$$\int_{\Omega}\{pW(\nabla\boldsymbol{y})-\Phi_{\boldsymbol{y}}(\boldsymbol{y})\cdot\boldsymbol{y}\}d\boldsymbol{x}=0, \tag{3.23}$$

$$\int_{\Omega}\{(n-p)W(\nabla\boldsymbol{y})-n\Phi(\boldsymbol{y})\}d\boldsymbol{x}=(1-p)\int_{\partial\Omega}W(\boldsymbol{a}\otimes\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{x})dS. \tag{3.24}$$

We can use (3.23) to eliminate $\int_{\Omega}W(\nabla\boldsymbol{y})$ from (3.24):

$$\int_{\Omega}\{(n-p)\Phi_{\boldsymbol{y}}(\boldsymbol{y})\cdot\boldsymbol{y}-np\Phi(\boldsymbol{y})\}d\boldsymbol{x}=p(1-p)\int_{\partial\Omega}W(\boldsymbol{a}\otimes\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{x})dS.$$

By assumption, $W(\boldsymbol{a}(\boldsymbol{x})\otimes\boldsymbol{n}(\boldsymbol{x}))\geq 0$ for all $\boldsymbol{x}\in\partial\Omega$, while on a star-shaped domain we have $\boldsymbol{n}\cdot\boldsymbol{x}\geq 0$ for all $\boldsymbol{x}\in\partial\Omega$. Therefore,

$$\int_{\Omega}\{np\Phi(\boldsymbol{y}(\boldsymbol{x}))-(n-p)\Phi_{\boldsymbol{y}}(\boldsymbol{y}(\boldsymbol{x}))\cdot\boldsymbol{y}(\boldsymbol{x})\}d\boldsymbol{x}\geq 0. \tag{3.25}$$

If the potential $\Phi(\boldsymbol{y})$ is such that the integrand in (3.25) is negative, except at $\boldsymbol{y}=0$, then $\boldsymbol{y}=0$ is the only smooth solution of the boundary value problem (3.17). Hence, we have proved the following theorem.

THEOREM **3.3.** *Suppose that $W\in C^1(\mathbb{R}^{m\times n})$ is a positively p-homogeneous function, $p\geq 1$, that is nonnegative on the rank-one cone. Let $\Phi\in C^1(\mathbb{R}^m)$ be such that $\Phi(0)=0$ and*

$$np\Phi(\boldsymbol{y})\leq(n-p)\Phi_{\boldsymbol{y}}(\boldsymbol{y})\cdot\boldsymbol{y} \tag{3.26}$$

for all $\boldsymbol{y}\in\mathbb{R}^m$. Then $\boldsymbol{y}(\boldsymbol{x})=0$ is the only C^2 solution of the boundary value problem (3.17), provided $\{0\}$ is the connected component of $Z=\{\boldsymbol{y}\in\mathbb{R}^m : np\Phi(\boldsymbol{y})=(n-p)\Phi_{\boldsymbol{y}}(\boldsymbol{y})\cdot\boldsymbol{y}\}$ that contains $\{0\}$.

We note that if $\Phi(\boldsymbol{y})$ is positively k-homogeneous, and $\Phi(\boldsymbol{y})>0$ for all $\boldsymbol{y}\neq 0$, then $\Phi(\boldsymbol{y})$ satisfies (3.26) whenever

$$\frac{1}{n}+\frac{1}{k}\leq\frac{1}{p}.$$

Indeed,

$$np\Phi(\boldsymbol{y}) - (n-p)\Phi_{\boldsymbol{y}}(\boldsymbol{y}) \cdot \boldsymbol{y} = \Phi(\boldsymbol{y})(np - (n-p)k) \leq 0.$$

Therefore, the conclusion of Theorem 3.3 applies when

$$\frac{1}{n} + \frac{1}{k} < \frac{1}{p}.$$

Note that in the critical case of equality in (3.26), $Z = \mathbb{R}^m$, and we must have

$$(1-p)W\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{n}} \otimes \boldsymbol{n}\right)(\boldsymbol{n} \cdot \boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega.$$

If $p > 1$, the domain is strictly star-shaped and W is positive on all the rank-one matrices, then we also have $\partial \boldsymbol{y}/\partial \boldsymbol{n} = 0$ on $\partial\Omega$. In may cases, as in the original Pohozaev's PDE

$$\Delta u + a^2 u^{\frac{n+2}{n-2}} = 0,$$

this is sufficient to assert that $\boldsymbol{y} = 0$ is the only solution.

4 Generalized Clapeyron theorem

In the previous section we presented several examples of variational problems with partial invariance and the correspondence between partial symmetries and partial conservation laws is exemplified by the classical Clapeyron's Theorem of linear elasticity. Here we show that it is but one member of a broader class of formulas expressing the stored elastic energy of the body in stable equilibrium in terms of physical and configurational tractions on its boundary. In what follows we show how different integral relations of this kind emerge as we vary our assumptions regarding the partial symmetries of the Lagrangian. We refer to a particularly important formula from this class associated with scale invarince of an associated variational problem as the Generalized Clapeyron Theorem (GCT).

Assume that $\boldsymbol{y}(\boldsymbol{x})$ satisfies all assumptions of Theorem 2.7, and consider a special class of scaling transformations of the form

$$\widetilde{\boldsymbol{x}} = e^{\epsilon}\boldsymbol{x}, \quad \widetilde{\boldsymbol{y}} = e^{\epsilon}\boldsymbol{y}. \tag{4.1}$$

In this case $\delta\boldsymbol{x} = \boldsymbol{x}$, $\delta\boldsymbol{y} = \boldsymbol{y}$, and $\widetilde{\boldsymbol{y}}(\widetilde{\boldsymbol{x}}) = e^{\epsilon}\boldsymbol{y}(\boldsymbol{x}) = e^{\epsilon}\boldsymbol{y}(e^{-\epsilon}\widetilde{\boldsymbol{x}})$. We compute

$$\delta E = \frac{d}{d\epsilon}\int_{\Omega} W(e^{\epsilon}\boldsymbol{x}, e^{\epsilon}\boldsymbol{y}(\boldsymbol{x}), \nabla\boldsymbol{y}(\boldsymbol{x}))e^{n\epsilon}d\boldsymbol{x}\bigg|_{\epsilon=0} = nE[\boldsymbol{y}] + \int_{\Omega}\{W_{\boldsymbol{x}} \cdot \boldsymbol{x} + W_{\boldsymbol{y}} \cdot \boldsymbol{y}(\boldsymbol{x})\}d\boldsymbol{x}. \tag{4.2}$$

By our assumptions, $\mathfrak{E}_W = 0$ and $\mathfrak{E}^*_W = -p^*_{\Sigma}\boldsymbol{n}\delta_{\Sigma},$. Therefore, the generalized Noether's formula (2.46) gives the following representation of the elastic energy

$$nE[\boldsymbol{y}] = -\int_{\Omega}\{W_{\boldsymbol{x}} \cdot \boldsymbol{x} + W_{\boldsymbol{y}} \cdot \boldsymbol{y}(\boldsymbol{x})\}d\boldsymbol{x} + \int_{\partial\Omega}\{\boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n} \cdot \boldsymbol{x}\}dS - \int_{\Sigma} p^*_{\Sigma}\boldsymbol{n} \cdot \boldsymbol{x}\, dS, \tag{4.3}$$

where $-W_{\boldsymbol{y}}$ has the meaning of externally applied bulk forces, while $-W_{\boldsymbol{x}}$ describes external configurational bulk forces. Equation (4.3) represents the most general Clapeyron-type relation which we use below to derive our main results.

4.1 Scale-free theories

Suppose further that the simultaneous change of scale $\boldsymbol{x} \mapsto \lambda\boldsymbol{x}$, $\boldsymbol{y} \mapsto \lambda\boldsymbol{y}$, with $\lambda > 0$, does not affect the energy density. In other words, we assume that there exists a partial symmetry of the form

$$W(\lambda\boldsymbol{x}, \lambda\boldsymbol{y}, \boldsymbol{F}) = W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}), \quad \forall\lambda > 0. \tag{4.4}$$

If W is continuous on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$, property (4.4) simply means that $W = W(\boldsymbol{F})$ and does not depend on $\boldsymbol{x}$ and $\boldsymbol{y}$ explicitly. If W is not required to be continuous at $\boldsymbol{x} = 0$ or $\boldsymbol{y} = 0$, then the class of functions satisfying (4.4) is much broader. In this case assumption (4.4) implies that

$$W_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}) \cdot \boldsymbol{x} + W_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}) \cdot \boldsymbol{y} = 0$$

identically. Hence, we have proved the following theorem

THEOREM **4.1.** *Suppose that $\Sigma \subset \Omega$ is a smooth surface of jump discontinuity of a Lipschitz extremal $\boldsymbol{y} \in C^2(\overline{\Omega} \setminus \Sigma)$ of the functional (2.1). If the energy density function satisfies (4.4), then*

$$nE[\boldsymbol{y}] = \int_{\partial\Omega} \{\boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n} \cdot \boldsymbol{x}\}dS - \int_{\Sigma} p^*_{\Sigma}\boldsymbol{n} \cdot \boldsymbol{x}\, dS. \tag{4.5}$$

If $\boldsymbol{y}$ is also stationary, i.e. $p^*_{\Sigma} = 0$, then

$$E[\boldsymbol{y}] = \frac{1}{n}\int_{\partial\Omega} \{\boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n} \cdot \boldsymbol{x}\}dS(\boldsymbol{x}). \tag{4.6}$$

Note that (4.6) is a generalization to not necessarily smooth extremals and arbitrary values of m and n of the known results in elasticity theory ($m = n = 3$) first obtained in [29, Eq. (16)] and independently in [39, formula (2.10)] (see also [57, Eq. (2.5)] and [47, Eq. (38)]).

We now turn to the interpretation of the obtained result. Observe first that due to the presence of the configurational stress $\boldsymbol{P}^*$ and the prefactor $1/n$, the relation (4.6) cannot be regarded as a simple "energy = work" relation, even though the first term seems to be very similar to the work of tractions $\boldsymbol{P}\boldsymbol{n}$ over displacements $\boldsymbol{u} = \boldsymbol{y} - \boldsymbol{x}$ on the boundary of the domain, since we can rewrite (4.6) as

$$E[\boldsymbol{y}] = \frac{1}{n}\int_{\partial\Omega} \{\boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{u} + \boldsymbol{P}^*_{\mathrm{disp}}\boldsymbol{n} \cdot \boldsymbol{x}\}dS(\boldsymbol{x}),$$

where

$$\boldsymbol{P}^*_{\mathrm{disp}} = W\boldsymbol{I}_n - (\nabla\boldsymbol{u})^T\boldsymbol{P}.$$

Note also that (4.6) is clearly fundamentally different from the classical Clapeyron's theorem (1.1), where the additive term on the right-hand side with the Eshelby (Hamilton, energy-momentum) tensor $\boldsymbol{P}^*$ is absent and even in 3D space instead of $1/n$ the multiple in front of the surface integral is $1/2$. As we are going to see in what follows the coefficient 1/2 emerges when we impose an additional assumption that the energy is a quadratic function of deformation gradient. The origin of the Eshelby stress in (4.6) is more interesting. It reflects a subtle fact that an elastic body may have a nonzero energy even if it is not loaded either internally by "physical" bulk forces or externally by "physical" surface tractions.

We recall that in the previous section we have already discussed a situation of this type in our Example 2.6. Using Theorem 4.1 we can generalize it to shed more light on what the loading by configurational tractions $\boldsymbol{P}^*\boldsymbol{n}$ is all about. Suppose that there exists a "transformation strain" $\boldsymbol{\varepsilon}_0(\widehat{\boldsymbol{x}})$, such that the energy density $W(\widehat{\boldsymbol{x}},\boldsymbol{\varepsilon})$ satisfies, for any $\boldsymbol{\varepsilon}\in\mathrm{Sym}(\mathbb{R}^n)$ and any unit vector $\widehat{\boldsymbol{x}}$, an inequality

$$W(\widehat{\boldsymbol{x}},\boldsymbol{\varepsilon})\geq c|\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0(\widehat{\boldsymbol{x}})|^p, \tag{4.7}$$

where $p>0$ and $c>0$. Note that the energy density (2.33) with (2.35) in our Example 2.6 is of this type.

Now, if there exists a Lipschitz stationary equilibrium $\boldsymbol{u}_0$ in Ω satisfying both $\boldsymbol{P}\boldsymbol{n}=0$ and $\boldsymbol{P}^*\boldsymbol{n}=0$ on $\partial\Omega$ in the sense of traces, then, according to (4.6) we must have

$$n\int_\Omega W(\widehat{\boldsymbol{x}},e(\boldsymbol{u}_0))d\boldsymbol{x}=0.$$

But then

$$\int_\Omega|e(\boldsymbol{u}_0)-\boldsymbol{\varepsilon}_0(\widehat{\boldsymbol{x}})|^p d\boldsymbol{x}\leq\frac{1}{c}\int_\Omega W(\widehat{\boldsymbol{x}},e(\boldsymbol{u}_0))d\boldsymbol{x}=0.$$

It would then mean that $\boldsymbol{\varepsilon}_0(\widehat{\boldsymbol{x}})=e(\boldsymbol{u}_0)$ is a compatible strain.

To make clear the connection between the incompatibility of $\boldsymbol{\varepsilon}_0(\widehat{\boldsymbol{x}})$ and the configurational tractions on the boundary we now rephrase the problem. Suppose that $\boldsymbol{\varepsilon}_0(\widehat{\boldsymbol{x}})$ is an incompatible strain field, and suppose that $\boldsymbol{u}_0$ is the minimizer in

$$\mathscr{M}=\min_{\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^n)}\int_\Omega W(\widehat{\boldsymbol{x}},e(\boldsymbol{u}))d\boldsymbol{x}. \tag{4.8}$$

Observe that due to (4.7), we can regard $\mathscr{M}>0$ as a measure of incompatibility of $\boldsymbol{\varepsilon}_0(\widehat{\boldsymbol{x}})$. At the same time, the minimizer $\boldsymbol{u}_0$ in (4.8) must be a stationary extremal, satisfying $\boldsymbol{P}\boldsymbol{n}=0$. But then, Theorem 4.1 gives

$$\mathscr{M}=\frac{1}{n}\int_{\partial\Omega}\boldsymbol{P}^*\boldsymbol{n}\cdot\boldsymbol{x}\,dS,$$

which ultimately links the "applied" configurational tractions to the incompatibility of the "transformation strain" $\boldsymbol{\varepsilon}_0(\widehat{\boldsymbol{x}})$.

Since in our Example 2.6 we could obtain the solution of the energy minimization problem explicitly, we could also directly compute the stored elastic energy which turned out to be different from zero despite the absence of "physical" loading. Here we independently compute the same energy by applying Theorem 4.1; it is obviously applicable because the energy denisity (2.33) with (2.35) satisfies (4.4). We obtain

$$E[\boldsymbol{u}]=\frac{1}{n}\int_{\partial\Omega}\boldsymbol{P}^*\boldsymbol{n}\cdot\boldsymbol{x}\,dS=\frac{1}{n}\int_{\partial\Omega}W(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F})\boldsymbol{n}\cdot\boldsymbol{x}\,dS. \tag{4.9}$$

where we used the fact that in our example "physical" tractions were absent and therefore $\boldsymbol{P}^*\boldsymbol{n}=W(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F})\boldsymbol{n}$. Furthermore, in our special case formula (4.9) gives

$$E[\boldsymbol{u}]=\frac{R}{n}\int_{\partial B(0,R)}W(\boldsymbol{x},\nabla\boldsymbol{y})dS=\frac{R|\partial B(0,R)|}{2n}\left((\eta'(R)-a)^2+(n-1)\frac{\eta(R)^2}{R^2}\right).$$

Using the explicit solution

$$\eta(r) = \frac{a}{n}\left(r + (n-1)r\ln\left(\frac{r}{R}\right)\right),$$

we can then obtain the expression for the energy

$$E[\boldsymbol{u}] = \frac{a^2(n-1)}{2n^2}|B(0,R)|,$$

which expectedly agrees with the rescaled energy (2.43).

We have therefore illustrated the fact that even if a body is unloaded externally (in view of the absence of physical forces) it can be still loaded externally by "applied" configurational forces. Such loading would then reveal itself through a residual prestress generated by the acquired incompatibility. We reiterate that such an incompatibility should be viewed as being effectively embedded into the body during the process of its creation, which can be in turn viewed as an outcome of the the work of configurational forces $\boldsymbol{P}^*\boldsymbol{n}$ applied on the boundary of the body. Specifically, in our Example 2.6 a nonzero configurational loading was represented by the configurational tractions

$$\boldsymbol{P}^*\boldsymbol{n} = \frac{a^2(n-1)}{2n^2}\boldsymbol{n} \neq 0.$$

We reiterate that the classical Clapeyron's Theorem does not account for such "work of creation" since it cannot accommodate any transformation strain as it breaks the homogeneity of degree 2, required for classical Clapeyron to hold. Instead, our GCT remains suitable because of the scale-invariant nature of the transformation strain in our Example 2.6.

We finish this section with a remark that will prove useful in what follows.

Remark 4.2. *When W does not depend explicitly on $\boldsymbol{x}$ and $\boldsymbol{y}$, the stationary extremals satisfy*

$$\int_{\partial\Omega} \boldsymbol{P}\boldsymbol{n}\, dS(\boldsymbol{x}) = \boldsymbol{0}, \qquad \int_{\partial\Omega} \boldsymbol{P}^*\boldsymbol{n}\, dS(\boldsymbol{x}) = \boldsymbol{0}.$$

These relations allow one to re-write the Clapeyron Theorem 4.1 as

$$\int_{\Omega} W(\nabla\boldsymbol{y})d\boldsymbol{x} = \frac{1}{n}\int_{\partial\Omega}\{(\boldsymbol{P}^*\boldsymbol{n}, \boldsymbol{x}-\boldsymbol{a}) + (\boldsymbol{P}\boldsymbol{n}, \boldsymbol{y}(\boldsymbol{x})-\boldsymbol{b})\}dS(\boldsymbol{x}), \tag{4.10}$$

where $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^m$ are arbitrary constant vectors.

4.2 Problems with constrants

We now show that the relation (4.6) can be easily generalized to cover the case of scale-free energy with scale-free pointwise constraints. A typical example is incompressible elasticity, where every deformation $\boldsymbol{y}(\boldsymbol{x})$ must satisfy the incompressibility constraint

$$\det\nabla\boldsymbol{y}(\boldsymbol{x}) = 1 \tag{4.11}$$

for all $\boldsymbol{x} \in \Omega$. Specifically, one can prove the following theorem:

THEOREM **4.3.** *Suppose that the deformation $\nabla \boldsymbol{y}(\boldsymbol{x})$ satisfies $i = 1,2,3...k$ pointwise constraints*

$$C^i(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})) = 0, \tag{4.12}$$

where both functions $W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})$ and $C^i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})$ are scale-free in the sense of (4.4). Then, if $\boldsymbol{y}(\boldsymbol{x})$ is a C^1 energy minimizer that satisfies the constraints, the following relation holds

$$\int_\Omega nW(\boldsymbol{x}, \boldsymbol{y}, \nabla \boldsymbol{y}) d\boldsymbol{x} = \int_{\partial\Omega} \{(\boldsymbol{P}_W - \Lambda_i(\boldsymbol{x})\boldsymbol{P}_{C^i})\boldsymbol{n} \cdot \boldsymbol{y} + (\boldsymbol{P}_W^* - \Lambda_i(\boldsymbol{x})\boldsymbol{P}_{C^i}^*)\boldsymbol{n} \cdot \boldsymbol{x}\} dS, \tag{4.13}$$

where $\Lambda_i(\boldsymbol{x})$ are the Lagrange multipliers corresponding to the constraints (4.12).

Proof. From the general theory of constrained variational problems we know that the C^1 energy minimizers can be obtained as stationary extremals of the augmented Lagrangian

$$L(\boldsymbol{x}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{F}}) = W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}) - \boldsymbol{\Lambda}(\boldsymbol{x}) \cdot \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}),$$

Here the functions $\boldsymbol{\Lambda}(\boldsymbol{x})$ are Lagrange multipliers, $\hat{\boldsymbol{y}} = [\boldsymbol{y}; \boldsymbol{\Lambda}] \in \mathbb{R}^{m+k}$, $\widehat{\boldsymbol{F}} = [\boldsymbol{F}; \boldsymbol{G}] \in \mathbb{R}^{(m+k)\times n}$ and $\boldsymbol{G} = \nabla \boldsymbol{\Lambda}$. Note first that the augmented Lagrangian is affine in $\boldsymbol{\Lambda}$ and does not depend explicitly on $\boldsymbol{G}$.

In the regular case, when $\boldsymbol{y}(\boldsymbol{x})$ is the energy minimizer satisfying the constraints, there are smooth Lagrange multipliers $\Lambda_i(\boldsymbol{x})$, such that $\hat{\boldsymbol{y}} = [\boldsymbol{y}; \boldsymbol{\Lambda}]$ solves the Euler-Lagrange equations $\mathfrak{E}_L(\boldsymbol{q}) = 0$ for the augmented Lagrangian, and hence also the stationarity equations $\mathfrak{E}_L^*(\boldsymbol{q}) = 0$.

Note next that by assumption both the Lagrangian and the constraints are scale-free. Therefore we can again consider the action of the following relevant deformation

$$\boldsymbol{X} = e^\epsilon \boldsymbol{x}, \quad \widehat{\boldsymbol{Y}} = [e^\epsilon \boldsymbol{y}; \Lambda].$$

The deformed functional

$$E_\epsilon[\widehat{\boldsymbol{Y}}] = \int_{e^\epsilon\Omega} L(\boldsymbol{X}, \widehat{\boldsymbol{Y}}_\epsilon(\boldsymbol{X}), \nabla \widehat{\boldsymbol{Y}}_\epsilon(\boldsymbol{X})) d\boldsymbol{X}$$

becomes, after the change of variables $\widetilde{\boldsymbol{x}} = e^\epsilon \boldsymbol{x}$

$$E_\epsilon = \epsilon^{n\epsilon} \int_\Omega \{W(e^\epsilon \boldsymbol{x}, e^\epsilon \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})) - \boldsymbol{\Lambda}(\boldsymbol{x}) \cdot \boldsymbol{C}(e^\epsilon \boldsymbol{x}, e^\epsilon \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x}))\} d\boldsymbol{x}.$$

and since both W and $\boldsymbol{C}$ satisfy (4.4), we have

$$E_\epsilon = \epsilon^{n\epsilon} \int_\Omega \{W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})) - \boldsymbol{\Lambda}(\boldsymbol{x}) \cdot \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x}))\} d\boldsymbol{x} = \epsilon^{n\epsilon} \int_\Omega W d\boldsymbol{x},$$

where we took into account that $\boldsymbol{y}(\boldsymbol{x})$ satisfies the constraints $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})) = 0$. Therefore,

$$\delta E = n \int_\Omega W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x})) d\boldsymbol{x},$$

and Noether identity (2.47) becomes (4.13), since $\delta \boldsymbol{x} = \boldsymbol{x}$, $\delta \widehat{\boldsymbol{y}} = [\boldsymbol{y}; 0]$,

$$\boldsymbol{P}_L = [\boldsymbol{P}_W - \Lambda_i(\boldsymbol{x})\boldsymbol{P}_{C^i}; 0], \quad \boldsymbol{P}_L^* = \boldsymbol{P}_W^* - \Lambda_i(\boldsymbol{x})\boldsymbol{P}_{C^i}^*.$$

□

In particular, when we deal with incompressible elasticity and the equilibrium configuration $\boldsymbol{y}(\boldsymbol{x})$ has to satisfy the constraint (4.11), the GCT reads

$$\int_{\Omega} W(\nabla \boldsymbol{y}(\boldsymbol{x}))d\boldsymbol{x} = \frac{1}{n}\int_{\partial\Omega}\{(\boldsymbol{P} - p(\boldsymbol{x})\mathrm{cof}\boldsymbol{F})\boldsymbol{n}\cdot\boldsymbol{y}(\boldsymbol{x}) + (\boldsymbol{P}^* + p(\boldsymbol{x})\boldsymbol{I}_n)\boldsymbol{n}\cdot\boldsymbol{x}\}dS(\boldsymbol{x}). \quad (4.14)$$

Here, following convention, we have denoted the Lagrange multiplier $\Lambda(\boldsymbol{x})$ by $p(\boldsymbol{x})$ to indicate its physical meaning as pressure, and took into account that

$$\boldsymbol{P}_C^* = (\det\boldsymbol{F} - 1)\boldsymbol{I}_n - \boldsymbol{F}^T\mathrm{cof}(\boldsymbol{F}) = -\boldsymbol{I}_n.$$

Note that when $\boldsymbol{y}(\boldsymbol{x})$ is an orientation preserving diffeomorphism, we can change variables $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$ in the first term on the right-hand side of (4.14) and obtain

$$\int_{\Omega} W(\nabla \boldsymbol{y})d\boldsymbol{x} = \frac{1}{n}\int_{\partial\Omega^*}(\boldsymbol{\sigma} - p(\boldsymbol{y})\boldsymbol{I}_n)\boldsymbol{N}\cdot\boldsymbol{y}\,dS^*(\boldsymbol{y}) + \frac{1}{n}\int_{\partial\Omega}(\boldsymbol{P}^* + p(\boldsymbol{x})\boldsymbol{I}_n)\boldsymbol{n}\cdot\boldsymbol{x}\,dS(\boldsymbol{x}), \quad (4.15)$$

where $\boldsymbol{\sigma}(\boldsymbol{y})$ is the Cauchy stress tensor, $\boldsymbol{N}(\boldsymbol{y})$ is the outward unit normal to $\partial\Omega^* = y(\partial\Omega)$, and $p(\boldsymbol{y}) = p(\boldsymbol{x})$, with $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$.

4.3 p-homogeneous energies

Now assume that the Lagrangian W instead of scale invariance has the following property of p-homogeneity

$$W(\boldsymbol{x}, \lambda\boldsymbol{y}, \lambda\boldsymbol{F}) = \lambda^p W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}). \quad (4.16)$$

In this case the deformation

$$\widetilde{\boldsymbol{x}} = \boldsymbol{x}, \qquad \widetilde{\boldsymbol{y}} = e^{\epsilon}\boldsymbol{y} \quad (4.17)$$

results in $\widetilde{\boldsymbol{y}}(\widetilde{\boldsymbol{x}}) = e^{\epsilon}\boldsymbol{y}(\boldsymbol{x}) = e^{\epsilon}\boldsymbol{y}(\widetilde{\boldsymbol{x}})$, so that

$$E[\widetilde{\boldsymbol{y}}](\epsilon) = \int_{\Omega} W(\widetilde{\boldsymbol{x}}, e^{\epsilon}\boldsymbol{y}(\widetilde{\boldsymbol{x}}), e^{\epsilon}\nabla\boldsymbol{y}(\widetilde{\boldsymbol{x}}))d\widetilde{\boldsymbol{x}} = e^{p\epsilon}E[\boldsymbol{y}].$$

We have now $\delta\boldsymbol{x} = 0$, $\delta\boldsymbol{y} = \boldsymbol{y}$ which leads to a new Clapeyron-type relation satisfied by equilibrium energy

$$\int_{\Omega} W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla\boldsymbol{y}(\boldsymbol{x}))d\boldsymbol{x} = \frac{1}{p}\int_{\partial\Omega}\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{y}\,dS. \quad (4.18)$$

We stress that the relation (4.18) is an expression of the p-homogeneity of the energy density that manifests itself through a rescaling of the displacements $\boldsymbol{y}$ without the concommittant rescaling of the Lagrangian coordinates. Therefore the validity of (4.18) does not require that $\boldsymbol{y}$ be stationary, i.e. solving (2.12). Also, since $\delta\boldsymbol{x} = 0$ the relation (4.18) remains valid even when the Lagrangian $W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})$ as a function of $\boldsymbol{x}$ is only measurable. We note that while (4.18) can be interpreted as a direct generalization of the classical Clapeyron's theorem because the latter is clearly a special case of the former, we preserve the name GCT for (4.6) as it marks a more radical generalization of this classical result linking together the work of physical and configurational forces.

Thus, below we show that a whole family of GCT type results emerge when scale invariance coexist with p-homogeneity. In particular we show that if the energy gets rescaled when we stretch the Lagrangian coordinates, without the concommittant stretching of the displacements,

$$W(\lambda^{-1}\boldsymbol{x}, \boldsymbol{y}, \lambda\boldsymbol{F}) = \lambda^p W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}), \tag{4.19}$$

then the energy can be expressed as a surface integral involving only the conjugate momenta $(\boldsymbol{P}^*)^T\boldsymbol{x}$.

Indeed, in this case the transformation

$$\widetilde{\boldsymbol{x}} = e^\epsilon \boldsymbol{x}, \qquad \widetilde{\boldsymbol{y}} = \boldsymbol{y} \tag{4.20}$$

results in $\widetilde{\boldsymbol{y}}(\widetilde{\boldsymbol{x}}) = \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{y}(e^{-\epsilon}\widetilde{\boldsymbol{x}})$, so that

$$E[\widetilde{\boldsymbol{y}}](\epsilon) = \int_{e^\epsilon\Omega} W(\widetilde{\boldsymbol{x}}, \boldsymbol{y}(e^{-\epsilon}\widetilde{\boldsymbol{x}}), e^{-\epsilon}\nabla\boldsymbol{y}(e^{-\epsilon}\widetilde{\boldsymbol{x}}))d\widetilde{\boldsymbol{x}} =$$
$$e^{-p\epsilon}\int_{e^\epsilon\Omega} W(e^{-\epsilon}\widetilde{\boldsymbol{x}}, \boldsymbol{y}(e^{-\epsilon}\widetilde{\boldsymbol{x}}), \nabla\boldsymbol{y}(e^{-\epsilon}\widetilde{\boldsymbol{x}}))d\widetilde{\boldsymbol{x}} = e^{(n-p)\epsilon}E[\boldsymbol{y}].$$

Now we have $\delta\boldsymbol{x} = \boldsymbol{x}$, $\delta\boldsymbol{y} = 0$, and therefore

$$E[\boldsymbol{y}] = \int_\Omega W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla\boldsymbol{y}(\boldsymbol{x}))d\boldsymbol{x} = \frac{1}{n-p}\int_{\partial\Omega} \boldsymbol{P}^*\boldsymbol{n}\cdot\boldsymbol{x}\,dS, \tag{4.21}$$

if $n \neq p$, and

$$\int_{\partial\Omega} \boldsymbol{P}^*\boldsymbol{n}\cdot\boldsymbol{x}\,dS = 0, \tag{4.22}$$

if $n = p$.

More generally, it is clear that in the case when $W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})$ is both scale-free and p-homogeneous, in the sense of (4.4) and (4.16), respectively, then (4.19) holds as a consequence:

$$W(\lambda^{-1}\boldsymbol{x}, \boldsymbol{y}, \lambda\boldsymbol{F}) = W(\lambda^{-1}\boldsymbol{x}, \lambda\lambda^{-1}\boldsymbol{y}, \lambda\boldsymbol{F}) = \lambda^p W(\lambda^{-1}\boldsymbol{x}, \lambda^{-1}\boldsymbol{y}, \boldsymbol{F}) = \lambda^p W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}).$$

Thus both formulas (4.18) and (4.21) (or (4.22), if $n = p$) hold. Combining (4.18) and (4.21), when $n \neq p$, we also obtain the relation between the two terms on the right-hand side of (4.6):

$$\frac{1}{p}\int_{\partial\Omega} \boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{y}\,dS = \frac{1}{n-p}\int_{\partial\Omega} \boldsymbol{P}^*\boldsymbol{n}\cdot\boldsymbol{x}\,dS. \tag{4.23}$$

Note that in view of (4.23) equation (4.6) which does not contain p can be obtained as a linear combination of (4.18) and (4.21). We remark that formula (4.23) is valid for any subdomain of Ω and can therefore be expressed as an apparently new concervation law

$$\nabla\cdot\big(p(\boldsymbol{P}^*)^T\boldsymbol{x} - (n-p)\boldsymbol{P}^T\boldsymbol{y}\big) = 0, \tag{4.24}$$

corresponding to the full variational symmetry unifying scale-invariance and p-homogeneity and described by the transformation group $(\boldsymbol{x}, \boldsymbol{y}) \mapsto (e^{p\epsilon}\boldsymbol{x}, e^{(p-n)\epsilon}\boldsymbol{y})$, $\epsilon \in \mathbb{R}$.

It is instructive to present explicit examples of Lagrangians that are both scale-free and p-homogeneous. If W depends on all of its variables, it would have to satisfy

$$W(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F})=\left(\frac{|\boldsymbol{y}|}{|\boldsymbol{x}|}\right)^{p}W\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\frac{|\boldsymbol{x}|}{|\boldsymbol{y}|}\boldsymbol{F}\right),\quad \hat{\boldsymbol{x}}=\frac{\boldsymbol{x}}{|\boldsymbol{x}|},\ \hat{\boldsymbol{y}}=\frac{\boldsymbol{y}}{|\boldsymbol{y}|}.$$

If W does not depend explicitly either on $\boldsymbol{x}$ or on $\boldsymbol{y}$, then

$$W(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F})=W(\widehat{\boldsymbol{w}},\boldsymbol{F}),\qquad W(\widehat{\boldsymbol{w}},\lambda\boldsymbol{F})=\lambda^{p}W(\widehat{\boldsymbol{w}},\boldsymbol{F}),$$

for every $\lambda>0$, where $\boldsymbol{w}$ is either $\boldsymbol{x}$ or $\boldsymbol{y}$.

Let us now demonstrate that in some sense the p-homogeneous Clapeyron-type Theorem (4.18) implies the GCT represented by (4.6). This becomes clear if we realize that at their source, the transformations (2.4) treat the $(\boldsymbol{x},\boldsymbol{y})$-space $\mathfrak{X}$ as a single entity. Therefore, to see the implied relation we need to go back to the description of the problem in the extended space introduced in Section 3.1.

We recall that in this space the original Lagrangian $W(\boldsymbol{x},\boldsymbol{y},\boldsymbol{F})$ is replaced by the *graph*-Lagrangian

$$\mathscr{W}(\boldsymbol{z},\boldsymbol{\mathscr{F}})=W(\boldsymbol{z},\boldsymbol{F}_2\boldsymbol{F}_1^{-1})\det\boldsymbol{F}_1,$$

where $\boldsymbol{z}\in\mathbb{R}^{n+m}$, $\boldsymbol{\mathscr{F}}=[\boldsymbol{F}_1;\boldsymbol{F}_2]\in\mathbb{R}^{(m+n)\times n}$, and $\boldsymbol{F}_1$ is the upper $n\times n$ submatrix of $\boldsymbol{\mathscr{F}}$.

Now, if $\mathscr{W}$ does not depend explicitly on $\boldsymbol{z}$, i.e., $W=W(\boldsymbol{F})$, then $\mathscr{W}(\boldsymbol{\mathscr{F}})$ satisfies (4.16), with $p=n$. Therefore,

$$E[\boldsymbol{y}]=\mathscr{E}[\boldsymbol{z},D]=\frac{1}{n}\int_{\partial D}\boldsymbol{\mathscr{P}}\boldsymbol{n}\cdot\boldsymbol{z}(\boldsymbol{t})dS(\boldsymbol{t}).$$

Using formula (3.7) for $\boldsymbol{\mathscr{P}}$ we obtain

$$E[\boldsymbol{y}]=\frac{1}{n}\int_{\partial D}\{\boldsymbol{P}^{*}(\nabla_{\boldsymbol{x}}\boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t})))\mathrm{cof}(\nabla_{\boldsymbol{t}}\boldsymbol{\eta}(\boldsymbol{t}))\boldsymbol{\nu}(\boldsymbol{t})\cdot\boldsymbol{\eta}(\boldsymbol{t})+ \\ \boldsymbol{P}(\nabla_{\boldsymbol{x}}\boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t})))\mathrm{cof}(\nabla_{\boldsymbol{t}}\boldsymbol{\eta}(\boldsymbol{t}))\boldsymbol{\nu}(\boldsymbol{t})\cdot\boldsymbol{y}(\boldsymbol{\eta}(\boldsymbol{t}))\}dS(\boldsymbol{t}),$$

where $\boldsymbol{\nu}(\boldsymbol{t})$ is the outward unit normal to ∂D. Changing variables in the above surface integral, using the relation

$$\mathrm{cof}(\nabla\boldsymbol{\eta}(\boldsymbol{t}))\boldsymbol{\nu}(\boldsymbol{t})dS(\boldsymbol{t})=\boldsymbol{n}(\boldsymbol{x})dS(\boldsymbol{x}),$$

we obtain the scale-invariant GCT represented by (4.6).

4.4 Case study: shock waves

The goal of this section is to apply Clapeyron's Theorem 4.1 (4.6) to dynamics where shock waves (dissipative jump discontinuities) emerge generically, and where the Lagrangian $L(\boldsymbol{\mathscr{F}})$ in the action functional $A[\boldsymbol{y}]$, given by (2.49) is scale-free because L does not depend explicitly on $\boldsymbol{q}=(t,\boldsymbol{x})$ and $\boldsymbol{y}$. By the Clapeyron Theorem 4.1,

$$\int_{\Omega^{n+1}}L(\boldsymbol{\mathscr{F}})d\boldsymbol{q}=\frac{1}{n+1}\int_{\partial\Omega^{n+1}}\{\boldsymbol{\mathscr{P}}\boldsymbol{N}\cdot\boldsymbol{y}+\boldsymbol{\mathscr{P}}^{*}\boldsymbol{N}\cdot\boldsymbol{q}\}d\mathcal{S}-\int_{\Sigma}\mathscr{P}^{*}_{\Sigma}(\boldsymbol{q})\boldsymbol{N}\cdot\boldsymbol{q}d\mathcal{S}.\qquad(4.25)$$

Our goal is to rewrite the "space-time" relation (4.25) using the conventional "space+time" decomposition. In view of (2.52), (2.55), and (2.68) we can write

$$\int_{\partial\Omega^{n+1}} \boldsymbol{\mathscr{P}}\boldsymbol{N}\cdot\boldsymbol{y}d\mathcal{S} = -\int_0^T\int_{\partial\Omega(t)}(V_n\boldsymbol{v}\cdot\boldsymbol{y}+\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{y})dSdt + \int_{\Omega(t)}\boldsymbol{v}\cdot\boldsymbol{y}d\boldsymbol{x}\Big|_0^T.$$

$$\int_{\partial\Omega^{n+1}} \boldsymbol{\mathscr{P}}^*\boldsymbol{N}\cdot\boldsymbol{q}d\mathcal{S} = \int_0^T\int_{\partial\Omega(t)}\{t(V_ne+\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{v})+V_n\boldsymbol{v}\cdot\boldsymbol{F}\boldsymbol{x}+\boldsymbol{Q}_L^*\boldsymbol{n}\cdot\boldsymbol{x}\}dSdt - \int_{\Omega(t)}\{te+\boldsymbol{F}\boldsymbol{x}\cdot\boldsymbol{v}\}d\boldsymbol{x}\Big|_0^T$$

where

$$\boldsymbol{Q}_L^* = L\boldsymbol{I}_n + \boldsymbol{F}^T\boldsymbol{P} = \frac{1}{2}|\boldsymbol{v}|^2\boldsymbol{I}_n - \boldsymbol{P}^*.$$

We also have

$$\int_\Sigma \mathscr{P}_\Sigma^*(\boldsymbol{q})\boldsymbol{N}\cdot\boldsymbol{q}d\mathcal{S} = \int_0^T\int_{S(t)}(\boldsymbol{x}\cdot\boldsymbol{n}^{\rm sh}-V^{\rm sh}t)\mathscr{P}_\Sigma^*dSdt.$$

Combining all these relations we obtain

$$\begin{aligned}(n+1)\int_0^T\int_{\Omega(t)}Ld\boldsymbol{x}dt = \int_{\Omega(t)}\{\boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})-te\}\,d\boldsymbol{x}\Big|_0^T + \\ \int_0^T\int_{\partial\Omega(t)}\{t(V_ne+\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{v})+V_n\boldsymbol{v}\cdot(\boldsymbol{F}\boldsymbol{x}-\boldsymbol{y})-\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{y}+\boldsymbol{Q}_L^*\boldsymbol{n}\cdot\boldsymbol{x}\}dSdt \\ -\int_0^T\int_{S(t)}(\boldsymbol{x}\cdot\boldsymbol{n}^{\rm sh}-V^{\rm sh}t)\mathscr{P}_\Sigma^*dSdt. \quad (4.26)\end{aligned}$$

Formula (4.26) is valid for any $T>0$. We can therefore differentiate it with respect to T:

$$\begin{aligned}(n+1)\int_{\Omega(t)}Ld\boldsymbol{x} = \frac{d}{dt}\int_{\Omega(t)}\{\boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})-te\}\,d\boldsymbol{x} \\ -\int_{\partial\Omega(t)}\{\boldsymbol{P}\boldsymbol{n}\cdot(\boldsymbol{y}-t\boldsymbol{v})-\boldsymbol{Q}_L^*\boldsymbol{n}\cdot\boldsymbol{x}-tV_ne-V_n\boldsymbol{v}\cdot(\boldsymbol{F}\boldsymbol{x}-\boldsymbol{y})+\}dS \\ -\int_{S(t)}(\boldsymbol{x}\cdot\boldsymbol{n}^{\rm sh}-V^{\rm sh}t)\mathscr{P}_\Sigma^*dS. \quad (4.27)\end{aligned}$$

We now use the previosuly obtained energy balance relation (2.69) to simplify (4.27)

$$\begin{aligned}\int_{\Omega(t)}\{(n+1)L+e\}d\boldsymbol{x} = \frac{d}{dt}\int_{\Omega(t)}\boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})d\boldsymbol{x} \\ -\int_{\partial\Omega(t)}\{\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{y}-\boldsymbol{Q}_L^*\boldsymbol{n}\cdot\boldsymbol{x}-V_n\boldsymbol{v}\cdot(\boldsymbol{F}\boldsymbol{x}-\boldsymbol{y})\}dS(\boldsymbol{x}) \\ -\int_{S(t)}\boldsymbol{x}\cdot\boldsymbol{n}^{\rm sh}\mathscr{P}_\Sigma^*dS(\boldsymbol{x}). \quad (4.28)\end{aligned}$$

Using the formula for the derivative of an integral over a moving volume and taking into account that the integrand has a jump discontinuity over a moving shock surface $S(t)$, we compute

$$\frac{d}{dt}\int_{\Omega(t)} \boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})d\boldsymbol{x} = \int_{\partial\Omega(t)} V_n \boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})dS + \int_{\Omega(t)} \left(\frac{\partial}{\partial t}\{\boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})\}\right)_{\text{reg}} d\boldsymbol{x} \\ - \int_{S(t)} V^{\text{sh}}[\![\boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})]\!]dS. \quad (4.29)$$

We also have

$$\left(\frac{\partial}{\partial t}\{\boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})\}\right)_{\text{reg}} = \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{\text{reg}} \cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x}) + \boldsymbol{v}\cdot(\boldsymbol{v}-(\nabla\boldsymbol{v})_{\text{reg}}\boldsymbol{x}) = \\ \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{\text{reg}} \cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x}) + |\boldsymbol{v}|^2 - \frac{1}{2}(\nabla|\boldsymbol{v}|^2)_{\text{reg}}\cdot\boldsymbol{x}$$

Integrating $(\nabla|\boldsymbol{v}|^2)_{\text{reg}}\cdot\boldsymbol{x}$ by parts, we obtain

$$\int_{\Omega(t)} \left(\frac{\partial}{\partial t}\{\boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})\}\right)_{\text{reg}} d\boldsymbol{x} = \int_{\Omega(t)} \left\{\left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{\text{reg}} \cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x}) + \frac{n+2}{2}|\boldsymbol{v}|^2\right\} d\boldsymbol{x} \\ - \int_{\partial\Omega(t)} \frac{1}{2}|\boldsymbol{v}|^2\boldsymbol{x}\cdot\boldsymbol{n}\,dS + \frac{1}{2}\int_{S(t)} [\![|\boldsymbol{v}|^2]\!]\boldsymbol{x}\cdot\boldsymbol{n}^{\text{sh}}dS.$$

Substituting this into (4.29) and back into (4.28), we obtain the desired dynamical version of the Clapeyron Theorem

$$n\int_{\Omega(t)} U d\boldsymbol{x} = \int_{\Omega(t)} \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{\text{reg}} \cdot(\boldsymbol{F}\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{x} + \int_{\partial\Omega(t)} \{\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n}\cdot\boldsymbol{x}\}dS(\boldsymbol{x}) \\ + \int_{S(t)} \{V^{\text{sh}}[\![\boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})]\!] - [\![\boldsymbol{v}]\!]\cdot\{\!\!\{\boldsymbol{v}\}\!\!\}(\boldsymbol{x}\cdot\boldsymbol{n}^{\text{sh}}) - p^*_{S(t)}\boldsymbol{x}\cdot\boldsymbol{n}^{\text{sh}}\}dS(\boldsymbol{x}),$$

where we have passed from the space-time $\mathscr{P}^*_\Sigma$ to its spatial version via (2.66). Next, using Hadamard's kinematic compatibility relations (2.63) on the shock in space-time we simplify the right-hand side of the relation above, by observing that

$$V^{\text{sh}}[\![\boldsymbol{v}\cdot(\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x})]\!] - [\![\boldsymbol{v}]\!]\cdot\{\!\!\{\boldsymbol{v}\}\!\!\}(\boldsymbol{x}\cdot\boldsymbol{n}^{\text{sh}}) = V^{\text{sh}}[\![\boldsymbol{v}]\!]\cdot(\boldsymbol{y}-\{\!\!\{\boldsymbol{F}\}\!\!\}\boldsymbol{x}),$$

and obtain our preliminary version of the dynamic Clapeyron Theorem

$$n\int_{\Omega(t)} U d\boldsymbol{x} = \int_{\Omega(t)} \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{\text{reg}} \cdot(\boldsymbol{F}\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{x} + \int_{\partial\Omega(t)} \{\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n}\cdot\boldsymbol{x}\}dS(\boldsymbol{x}) \\ + \int_{S(t)} \{V^{\text{sh}}[\![\boldsymbol{v}]\!]\cdot(\boldsymbol{y}-\{\!\!\{\boldsymbol{F}\}\!\!\}\boldsymbol{x}) - p^*_{S(t)}\boldsymbol{x}\cdot\boldsymbol{n}^{\text{sh}}\}dS(\boldsymbol{x}), \quad (4.30)$$

To associate physical meaning with different terms in the right-hand side of (4.30), we first introduce several notations. Thus, we observe that

$$\boldsymbol{b}_i = -\left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{\text{reg}}$$

is the regular component of the inertial (d'Alambert) body force density. The Noether identity (2.13) assigns to

$$\boldsymbol{b}_i^* = \boldsymbol{F}^T\left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{\text{reg}} = -\boldsymbol{F}^T\boldsymbol{b}_i$$

the meaning of the the regular component of the inertial configurational body force density.

The distributional part of the inertial force density $-\boldsymbol{a} = -\partial\boldsymbol{v}/\partial t$ is supported on the shock surface and can be also interperted as inertial traction on the shock; it takes the form

$$\boldsymbol{t}_i = V^{\text{sh}}[\![\boldsymbol{v}]\!].$$

Similar reasoning suggests that

$$\boldsymbol{t}_i^* = -V^{\text{sh}}\{\!\{\boldsymbol{F}\}\!\}^T[\![\boldsymbol{v}]\!] = -\{\!\{\boldsymbol{F}\}\!\}^T\boldsymbol{t}_i$$

can be identified with the configurational inertial tractions on the shock.

Using these notations, we can now present the dynamic GCT in the final form

$$\int_{\Omega(t)} U d\boldsymbol{x} = \frac{\mathfrak{S}+\mathfrak{I}}{n}, \tag{4.31}$$

where we separated in the right-hand side of (4.31) the "static" term $\mathfrak{S}$ from the "dynamic", or inertial term $\mathfrak{I}$. The former

$$\mathfrak{S} = \int_{\partial\Omega(t)}\{\boldsymbol{P}\boldsymbol{n}\cdot\boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n}\cdot\boldsymbol{x}\}dS(\boldsymbol{x}) - \int_{S(t)} p^*_{S(t)}\boldsymbol{x}\cdot\boldsymbol{n}^{\text{sh}}dS(\boldsymbol{x}). \tag{4.32}$$

represents the work of the physical and configurational forces on the boundary of the domain, minus the energy dissipated on the shock surface (see formula (4.5) in the statics case). The latter

$$\mathfrak{I} = \int_{\Omega(t)}\{\boldsymbol{b}_i\cdot\boldsymbol{y} + \boldsymbol{b}_i^*\cdot\boldsymbol{x}\}d\boldsymbol{x} + \int_{S(t)}\{\boldsymbol{t}_i\cdot\boldsymbol{y} + \boldsymbol{t}_i^*\cdot\boldsymbol{x}\}dS(\boldsymbol{x}) \tag{4.33}$$

represents the work of the inertial physical and configurational forces inside the domain, plus the work of the corresponding inertial tractions acting on the shock surface.

5 Applications

In this section we show the utility of the Clapeyron Theorem in various problems of mechanics and physics rooted in the Calculus of Variations. Throughout this section we assume that $\boldsymbol{y}(\boldsymbol{x})$ is a Lipschitz stationary extremal in Ω that is class C^2 on a neighborhood of $\partial\Omega$, as in Remark 2.8.

5.1 Necessary condition of metastability

The problem of metastability is central for nonlinear elasticity theory and, more generally, for vectorial variational problems with nonconvex energy density. In our special setting, it concerns the existence of strong local minimia that are not automatically global. To analyze this problem we can take advantage of the fact that GCT gives a formula for the energy in terms of the boundary values of the solution and its gradients. The idea is to consider the difference of two such expressions and rearrange it advantageously. As our subsequent analysis shows a helpful hint is to isolate in the boundary integral an expression involving a Weierstrass excess function

$$\mathscr{E}(\boldsymbol{F},\boldsymbol{G}) = W(\boldsymbol{G}) - W(\boldsymbol{F}) - (W_{\boldsymbol{F}}(\boldsymbol{F}), \boldsymbol{G}-\boldsymbol{F}), \tag{5.1}$$

and then compare the energies of two different configurations with the same Dirchlet data. The result of the implied rearrangement can be formulated in the form of a theorem which relies essentially on the GCT.

THEOREM **5.1.** *Assume that the Lipschitz deformations $\boldsymbol{y}_1(\boldsymbol{x})$ and $\boldsymbol{y}_2(\boldsymbol{x})$ are stationary, of class C^2 near $\partial\Omega$ and that*

$$\boldsymbol{y}_1(\boldsymbol{x}) = \boldsymbol{y}_2(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}), \quad \boldsymbol{x}\in\partial\Omega. \tag{5.2}$$

Then

$$\int_\Omega \{W(\boldsymbol{F}_2) - W(\boldsymbol{F}_1)\}d\boldsymbol{x} = \frac{1}{n}\int_{\partial\Omega}\{\mathscr{E}(\boldsymbol{F}_1,\boldsymbol{F}_2)(\boldsymbol{x},\boldsymbol{n}) + (\boldsymbol{P}_1-\boldsymbol{P}_2)\boldsymbol{n}\cdot(\boldsymbol{F}_2\boldsymbol{x}-\boldsymbol{y}_0)\}dS(\boldsymbol{x}), \tag{5.3}$$

where

$$\boldsymbol{F}_j = \nabla\boldsymbol{y}_j(\boldsymbol{x}), \qquad \boldsymbol{P}_j = W_{\boldsymbol{F}}(\nabla\boldsymbol{y}_j(\boldsymbol{x})), \qquad j=1,2.$$

Proof. Since the energy is scale-free, we can use the nonlinear Clapeyron Theorem (4.6) and write

$$\int_\Omega W(\boldsymbol{F}_i)d\boldsymbol{x} = \frac{1}{n}\int_{\partial\Omega}\{\boldsymbol{P}_i\boldsymbol{n}\cdot\boldsymbol{y}_0 + \boldsymbol{P}_i^*\boldsymbol{n}\cdot\boldsymbol{x}\}dS,$$

Subtracting these two equations and expanding $\boldsymbol{P}_j^*$ via (2.8) we obtain

$$\int_\Omega \{W(\boldsymbol{F}_2) - W(\boldsymbol{F}_1)\}d\boldsymbol{x} = \frac{1}{n}\int_{\partial\Omega}\{\mathscr{E}(\boldsymbol{F}_1,\boldsymbol{F}_2)(\boldsymbol{n}\cdot\boldsymbol{x}) + (\boldsymbol{P}_1-\boldsymbol{P}_2)\boldsymbol{n}\cdot(\boldsymbol{F}_2\boldsymbol{x}-\boldsymbol{y}_0) + \langle\boldsymbol{P}_1,\boldsymbol{F}_2-\boldsymbol{F}_1\rangle(\boldsymbol{n}\cdot\boldsymbol{x}) + (\boldsymbol{P}_1\boldsymbol{n},(\boldsymbol{F}_1-\boldsymbol{F}_2)\boldsymbol{x})\}dS.$$

Since $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are of class C^2 near $\partial\Omega$ and $\boldsymbol{y}_1-\boldsymbol{y}_2=0$ on $\partial\Omega$, we conclude that there exists a C^1 field $\boldsymbol{a}:\partial\Omega\to\mathbb{R}^m$, such that $\boldsymbol{F}_1-\boldsymbol{F}_2=\boldsymbol{a}\otimes\boldsymbol{n}$. Therefore,

$$(\boldsymbol{P}_1\boldsymbol{n},(\boldsymbol{F}_1-\boldsymbol{F}_2)\boldsymbol{x}) = (\boldsymbol{P}_1\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{n}\cdot\boldsymbol{x}) = \langle\boldsymbol{P}_1,\boldsymbol{F}_1-\boldsymbol{F}_2\rangle(\boldsymbol{n}\cdot\boldsymbol{x}),$$

and (5.3) follows. □

Remark 5.2. *Switching $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ we also have*

$$\int_\Omega \{W(\boldsymbol{F}_2) - W(\boldsymbol{F}_1)\}d\boldsymbol{x} = \frac{1}{n}\int_{\partial\Omega}\{-\mathscr{E}(\boldsymbol{F}_2,\boldsymbol{F}_1)(\boldsymbol{x},\boldsymbol{n}) + (\boldsymbol{P}_1-\boldsymbol{P}_2)\boldsymbol{n}\cdot(\boldsymbol{F}_1\boldsymbol{x}-\boldsymbol{y}_0)\}dS(\boldsymbol{x}). \tag{5.4}$$

Taking the average of (5.3) and (5.4) we obtain a more symmetric version of the energy increment formula

$$\int_{\Omega}\{W(\boldsymbol{F}_2)-W(\boldsymbol{F}_1)\}d\boldsymbol{x}=$$
$$\frac{1}{n}\int_{\partial\Omega}\left\{\frac{1}{2}(\mathscr{E}(\boldsymbol{F}_1,\boldsymbol{F}_2)-\mathscr{E}(\boldsymbol{F}_2,\boldsymbol{F}_1))(\boldsymbol{x},\boldsymbol{n})+(\boldsymbol{P}_1-\boldsymbol{P}_2,\{\!\!\{\boldsymbol{F}\}\!\!\}\boldsymbol{x}-\boldsymbol{y}_0)\right\}dS(\boldsymbol{x}),$$

where $\{\!\!\{\boldsymbol{F}\}\!\!\}=(\boldsymbol{F}_1+\boldsymbol{F}_2)/2$*. Subtracting (5.3) and (5.4) we also obtain*

$$\int_{\partial\Omega}(\mathscr{E}(\boldsymbol{F}_1,\boldsymbol{F}_2)+\mathscr{E}(\boldsymbol{F}_2,\boldsymbol{F}_1))(\boldsymbol{x},\boldsymbol{n})dS(\boldsymbol{x})=\int_{\partial\Omega}((\boldsymbol{P}_2-\boldsymbol{P}_1)\boldsymbol{n},(\boldsymbol{F}_2-\boldsymbol{F}_1)\boldsymbol{x})dS(\boldsymbol{x}) \tag{5.5}$$

Observe further that if $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, satisfying (5.2), are smooth near $\partial\Omega$, then $\boldsymbol{F}_2-\boldsymbol{F}_1=\boldsymbol{a}(\boldsymbol{x})\otimes\boldsymbol{n}(\boldsymbol{x})$ for all $\boldsymbol{x}\in\partial\Omega$, where $\boldsymbol{n}(\boldsymbol{x})$ is an outward unit normal to $\partial\Omega$. Observe also that if $W(\boldsymbol{F})$ is rank-one convex, we have for all $\boldsymbol{x}\in\partial\Omega$

$$\mathscr{E}(\boldsymbol{F}_1,\boldsymbol{F}_2)=W(\boldsymbol{F}_2)-W(\boldsymbol{F}_1)-\langle\boldsymbol{P}_1,\boldsymbol{F}_2-\boldsymbol{F}_1\rangle\geq 0.$$

In view of these observations and the fact that, if the origin is a star point in Ω, which can always be arranged by a translation due to Remark 4.2, then $(\boldsymbol{x},\boldsymbol{n})\geq 0$ for all $\boldsymbol{x}\in\Omega$, we can conclude that whenever $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are two smooth extremals satisfying (5.2), the following inequality holds

$$\int_{\Omega}\{W(\boldsymbol{F}_2)-W(\boldsymbol{F}_1)\}d\boldsymbol{x}\geq\frac{1}{n}\int_{\partial\Omega}\{(\boldsymbol{P}_1-\boldsymbol{P}_2)\boldsymbol{n}\cdot(\boldsymbol{F}_2\boldsymbol{x}-\boldsymbol{y}_0)\}dS(\boldsymbol{x}). \tag{5.6}$$

Now, if $\boldsymbol{y}_2(\boldsymbol{x})$ is a global minimizer and $\boldsymbol{y}_1(\boldsymbol{x})$ is a stationary extremal satisfying

$$\int_{\partial\Omega}(\boldsymbol{P}_1-\boldsymbol{P}_2)\boldsymbol{n}\cdot(\boldsymbol{F}_2\boldsymbol{x}-\boldsymbol{y}_0)dS(\boldsymbol{x})\geq 0, \tag{5.7}$$

then

$$0\geq\int_{\Omega}\{W(\boldsymbol{F}_2)-W(\boldsymbol{F}_1)\}d\boldsymbol{x}\geq\int_{\partial\Omega}(\boldsymbol{P}_1-\boldsymbol{P}_2)\boldsymbol{n}\cdot(\boldsymbol{F}_2\boldsymbol{x}-\boldsymbol{y}_0)dS(\boldsymbol{x})\geq 0,$$

and hence $\boldsymbol{y}_1(\boldsymbol{x})$ must also be a global minimizer. One can see that to allow for metastaility condition (5.7) should be violated and since the left-hand side of (5.7) can be regarded as a measure of non-affinity of $\boldsymbol{y}_2(\boldsymbol{x})$, such non-affinity can be then viewed as a necessary condition of metastability [38].

5.2 Quasiconvex envelope

To formulate the problem considered in this section we would need first to recall several standard definitions. The most important one is the definition of the quasiconvex envelope of the energy density $W(\boldsymbol{F})$—the largest quasiconvex function that does not exceed $W(\boldsymbol{F})$. It is denoted QW and is often referred to as quasiconvexification of W. There is a formula for QW [21]

$$QW(\boldsymbol{F})|D|=\inf_{\substack{\boldsymbol{y}|_{\partial D}=\boldsymbol{F}\boldsymbol{x}\\ \boldsymbol{y}\in W^{1,\infty}(D;\mathbb{R}^m)}}\int_D W(\nabla\boldsymbol{y})d\boldsymbol{x}. \tag{5.8}$$

We would also need the notion of rank-one convexity, i.e. convexity along all straight lines connecting two points that differ by a rank-one matrix. Similarly to the quasiconvex envelope of W we define the rank-one convex envelope RW to be the largest rank-one convex function that does not exceed W. It was shown in [76] that every quasiconvex function is rank-one convex and therefore, $RW(\boldsymbol{F}) \geq QW(\boldsymbol{F})$. The question whether $QW(\boldsymbol{F}) = RW(\boldsymbol{F})$ has been answered in the negative in [104], except in the case $m = 2$. Nonetheless examples where $QW(\boldsymbol{F}) \neq RW(\boldsymbol{F})$ are extremely rare [35].

While, in many cases the variational problem (5.8) has no solutions, it is known that the quasiconvexification QW of W does not depend on the choice of the domain D in (5.8), and there are cases when one is be able to choose a particular domain D, where Lipschitz solutions do exist [38]. In those cases, the minimizers must be stationary extremals for which GCT is applicable.

In this context one can regard such minimizers as known and pose the question whether GCT provides information about the function QW in general, rather than about a specific solution of equilibrium equations. One representative result of this type, allowing one to transfer the knowledge about quasiconvexity from the boundary to the interior of the domain, is formulated below in the form of a theorem:

THEOREM **5.3.** *Let $\Omega \subset \mathbb{R}^d$ be a star-shaped domain with Lipschitz boundary. Assume that*

(BVP) For a given $\boldsymbol{F}_0 \in \mathbb{R}^{m\times n}$ the function $\overline{\boldsymbol{y}} \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ solves

$$\nabla \cdot \boldsymbol{P}(\nabla\overline{\boldsymbol{y}}) = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{P}^*(\nabla\overline{\boldsymbol{y}}) = \boldsymbol{0}, \tag{5.9}$$

in the sense of distributions, and satisfies $\overline{\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{F}_0\boldsymbol{x}$ for all $\boldsymbol{x} \in \partial\Omega$;

(REG) $\overline{\boldsymbol{y}}$ is of class C^2 near $\partial\Omega$;

(RCX) $W(\nabla\overline{\boldsymbol{y}}(\boldsymbol{x})) = RW(\nabla\overline{\boldsymbol{y}}(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \partial\Omega$;

(Q=R) $RW(\boldsymbol{F}_0) = QW(\boldsymbol{F}_0)$.

Then

(i) $RW(\boldsymbol{F}_0) = QW(\boldsymbol{F}_0) = ⨍_\Omega W(\nabla\overline{\boldsymbol{y}})d\boldsymbol{x}$, i.e. $\overline{\boldsymbol{y}}(\boldsymbol{x})$ is the global minimizer in (5.8).

(ii) $QW(\nabla\overline{\boldsymbol{y}}(\boldsymbol{x})) = W(\nabla\overline{\boldsymbol{y}}(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Omega$.

Proof. Assumption (BVP) implies that the Clapeyron theorem is applicable to the stationary equilibrium $\overline{\boldsymbol{y}}(\boldsymbol{x})$ of (5.8) we obtain, by virture of (REG),

$$\int_\Omega W(\nabla\overline{\boldsymbol{y}})d\boldsymbol{x} = \frac{1}{n}\int_{\partial\Omega}(W(\boldsymbol{F}_0+\boldsymbol{a}\otimes\boldsymbol{n}) - \boldsymbol{P}(\boldsymbol{F}_0+\boldsymbol{a}\otimes\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{n}\cdot\boldsymbol{x})dS. \tag{5.10}$$

Next, we appeal to [37, Lemma 4.2]. We quote the part of the lemma needed for the proof.

LEMMA **5.4.** *Let $V(\boldsymbol{F}_0)$ be a rank-one convex function such that $V(\boldsymbol{F}_0) \leq W(\boldsymbol{F}_0)$. Let*

$$\mathscr{A}_V = \{\boldsymbol{F}_0 \in \mathbb{R}^{m\times n} : W(\boldsymbol{F}_0) = V(\boldsymbol{F}_0)\}.$$

Then for every $\boldsymbol{F}_* \in \mathscr{A}_V$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{m} \in \mathbb{R}^n$

$$V(\boldsymbol{F}_* + \boldsymbol{b}\otimes\boldsymbol{m}) \geq W(\boldsymbol{F}_*) + \boldsymbol{P}(\boldsymbol{F}_*)\boldsymbol{m}\cdot\boldsymbol{b}. \tag{5.11}$$

We now apply Lemma 5.4 by choosing $V(\boldsymbol{F}_0) = RW(\boldsymbol{F}_0)$, $\boldsymbol{F}_* = \nabla\overline{\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{F}_0 + \boldsymbol{a}\otimes\boldsymbol{n}$, and $\boldsymbol{b}\otimes\boldsymbol{m} = -\boldsymbol{a}\otimes\boldsymbol{n}$, for each $\boldsymbol{x}\in\partial\Omega$. The assumption (RCX) implies that Lemma 5.4 is applicable, and therefore,

$$RW(\boldsymbol{F}_0) \geq W(\boldsymbol{F}_0 + \boldsymbol{a}\otimes\boldsymbol{n}) - \boldsymbol{P}(\boldsymbol{F}_0 + \boldsymbol{a}\otimes\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{a}. \tag{5.12}$$

Finally, we use the assumption that Ω is star-shaped. If we choose the origin at the star point, then the function $\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{x}$ is always non-negative at all points on $\partial\Omega$. Therefore, inequality (5.12) together with the identity (5.10) delivered by the Clapeyron theorem, implies

$$\int_\Omega W(\nabla\overline{\boldsymbol{y}})d\boldsymbol{x} \leq \frac{RW(\boldsymbol{F}_0)}{n}\int_{\partial\Omega}(\boldsymbol{n}\cdot\boldsymbol{x})dS = |\Omega|RW(\boldsymbol{F}_0).$$

Since $|\Omega|QW(\boldsymbol{F}_0)$ is the minimal value of the functional in (5.8) we obtain

$$|\Omega|RW(\boldsymbol{F}_0) \geq \int_\Omega W(\nabla\overline{\boldsymbol{y}})d\boldsymbol{x} \geq |\Omega|QW(\boldsymbol{F}_0). \tag{5.13}$$

By the assumption (Q=R) we must have equality in both inequalities in (5.13) and conclude that $\overline{\boldsymbol{y}}(\boldsymbol{x})$ is the global minimizer in (5.8), proving (i). Property (ii) is a necessary condition for any strong local minimizer [106], and hence must hold as well. $\square$

5.3 Probing the binodal

It is well-known [3] that if $\boldsymbol{y}(\boldsymbol{x})$ is a strong local minimizer of (2.1), then the function $\boldsymbol{F}\mapsto W(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x}),\boldsymbol{F})$ must be quasiconvex at $\boldsymbol{F}$, i.e. satisfy the inequality

$$\int_D \{W(\boldsymbol{F}+\nabla\boldsymbol{\phi}(\boldsymbol{x})) - W(\boldsymbol{F})\}d\boldsymbol{x} \geq 0 \quad \forall\boldsymbol{\phi}\in W_0^{1,\infty}(D;\mathbb{R}^m), \tag{5.14}$$

where D is any domain in $\mathbb{R}^n$. Thus, the gradient of any metastable configuration must avoid the binodal region of the phase space

$$\mathscr{B} = \left\{\boldsymbol{F}\in\mathbb{R}^{m\times n} : \exists\boldsymbol{\phi}\in W_0^{1,\infty}(D;\mathbb{R}^m) : \int_D\{W(\boldsymbol{F}+\nabla\boldsymbol{\phi}(\boldsymbol{x})) - W(\boldsymbol{F})\}d\boldsymbol{x} < 0\right\}.$$

Its boundary $\partial\mathscr{B}$ is called the *binodal*.

Since the function is quasiconvex, if and only if its binodal region is empty, the binodal region can then be characterized by the inequality [50]

$$\mathscr{B} = \{\boldsymbol{F}\in\mathbb{R}^{m\times n} : QW(\boldsymbol{F}) < W(\boldsymbol{F})\}.$$

Note also that if $W(\boldsymbol{F}_0)$ is strictly quasiconvex at $\boldsymbol{F}_0\in\mathbb{R}^{m\times n}$, then $\overline{\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{F}_0\boldsymbol{x}$ is the unique solution of (5.8) [107]. If $\boldsymbol{F}_0$ lies on the binodal, we expect the emergence of nontrivial equilibria in unbounded domains, where the volume fraction of the region where $\nabla\overline{\boldsymbol{y}}$ is a finite distance from $\boldsymbol{F}_0$ to be zero, even though in finite domains we still expect that $\overline{\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{F}_0\boldsymbol{x}$ be the unique solution. To follow this path one can, for instance, try to find radial solutions of (5.9) in the entire space. As we show below, in case of the success, Clapeyron's Theorem can deliver a formula for $QW(\boldsymbol{F})$ for a range of $\boldsymbol{F}$.

Suppose $m=n$, and $W(\boldsymbol{F})$ is objective and isotropic

$$W(\boldsymbol{F})=w(v_1,\ldots,v_n),$$

where v_j are the sigular values of $\boldsymbol{F}$ and the function w does not change after a permutation of its arguments. We look for a radial solution $\boldsymbol{y}(\boldsymbol{x})=\eta(r)\widehat{\boldsymbol{x}}$ of (5.9). We first observe that

$$\nabla\boldsymbol{y}=\eta'\widehat{\boldsymbol{x}}\otimes\widehat{\boldsymbol{x}}+\frac{\eta}{r}(\boldsymbol{I}_n-\widehat{\boldsymbol{x}}\otimes\widehat{\boldsymbol{x}}).$$

Hence, $\boldsymbol{F}=\nabla\boldsymbol{y}$ has a singular value $|\eta'(r)|$ and $n-1$ singular values $|\eta(r)/r|$. We will assume that $\eta(r)\ge 0$ and $\eta'(r)\ge 0$. In that case we define two functions

$$w_1(r)=\frac{\partial w}{\partial v_1}(\eta'(r),\eta(r)/r,\ldots,\eta(r)/r),\quad w_2(r)=\frac{\partial w}{\partial v_2}(\eta'(r),\eta(r)/r,\ldots,\eta(r)/r).$$

The Piola stress tensor is [4, p. 564]

$$\boldsymbol{P}=w_1\widehat{\boldsymbol{x}}\otimes\widehat{\boldsymbol{x}}+w_2(\boldsymbol{I}_n-\widehat{\boldsymbol{x}}\otimes\widehat{\boldsymbol{x}}).$$

Therefore, $\eta(r)$ must solve the nonlinear second order ODE

$$w_{11}\eta''+(n-1)w_{12}\left(\frac{\eta}{r}\right)'+\frac{n-1}{r}(w_1-w_2)=0, \tag{5.15}$$

where

$$w_{11}(r)=\frac{\partial^2 w}{\partial v_1^2}(\eta'(r),\eta(r)/r,\ldots,\eta(r)/r),\quad w_{12}(r)=\frac{\partial^2 w}{\partial v_1\partial v_2}(\eta'(r),\eta(r)/r,\ldots,\eta(r)/r).$$

If $\eta'(r)$ suffers a jump discontinuity at $r=r_0$, then for (5.9) to hold we must have

$$[\![w_1]\!](r_0)=0,\quad [\![w]\!](r_0)=w_1(r_0)[\![\eta']\!].$$

Typically, we would look for a solution of the form

$$\boldsymbol{y}(\boldsymbol{x})=\begin{cases}f_0r\widehat{\boldsymbol{x}}, & |\boldsymbol{x}|<1,\\ \eta(r)\widehat{\boldsymbol{x}}, & |\boldsymbol{x}|>1\end{cases} \tag{5.16}$$

In this case we must have

$$w_1(\eta'(1^+),f_0,\ldots,f_0)=w_1(f_0,f_0,\ldots,f_0)$$

$$w(\eta'(1^+),f_0,\ldots,f_0)=w(f_0,f_0,\ldots,f_0)+w_1(f_0,f_0,\ldots,f_0)(\eta'(1^+)-f_0).$$

These can be regarded as equations determining the values of f_0 and $\eta'(1^+)$. Ultimately, the radial solution $\eta(r)$ will be fixed by

$$f_\infty=\lim_{r\to\infty}\frac{\eta(r)}{r}=\lim_{r\to\infty}\eta'(r).$$

In Appendix A.1 we prove that

$$\eta(r) = f_\infty r + \frac{A}{r^{n-1}} + O(r^{-n}),\ \text{as } r \to \infty. \tag{5.17}$$

Next we show how Clapeyron's Theorem can use the knowledge of quasiconvexity at infinity (the boundary of all-space), transfers it to the interior and finally delivers an explicit formula for $QW(f\boldsymbol{I}_n)$ for all $f \in [f_0, f_\infty]$.

THEOREM **5.5.** *Suppose that $\eta \in C^2([1, +\infty))$ is such that*

(i) $\overline{\boldsymbol{y}}(\boldsymbol{x}) = \begin{cases} f_0\boldsymbol{x}, & |\boldsymbol{x}| < 1, \\ \eta(|\boldsymbol{x}|)\widehat{\boldsymbol{x}}, & |\boldsymbol{x}| > 1 \end{cases}$ *solves (5.9) in $\mathbb{R}^n$ in the sense of distributions.*

(ii) $f_\infty = \lim\limits_{r\to\infty} \dfrac{\eta(r)}{r} = \lim\limits_{r\to\infty} \eta'(r)$ *is such that $W(f_\infty\boldsymbol{I}_n) = QW(f_\infty\boldsymbol{I}_n)$.*

Then

(a) $W(f_0\boldsymbol{I}_n) = QW(f_0\boldsymbol{I}_n)$*;*

(b) for all $r \geq 1$

$$W\left(\eta'(r)\widehat{\boldsymbol{x}} \otimes \widehat{\boldsymbol{x}} + \frac{\eta(r)}{r}(\boldsymbol{I}_n - \widehat{\boldsymbol{x}} \otimes \widehat{\boldsymbol{x}})\right) = QW\left(\eta'(r)\widehat{\boldsymbol{x}} \otimes \widehat{\boldsymbol{x}} + \frac{\eta(r)}{r}(\boldsymbol{I}_n - \widehat{\boldsymbol{x}} \otimes \widehat{\boldsymbol{x}})\right). \tag{5.18}$$

(c) for every $r \geq 1$

$$QW\left(\frac{\eta(r)}{r}\boldsymbol{I}_n\right) = w_1(r)\frac{\eta(r)}{r} + w(r) - \eta'(r)w_1(r). \tag{5.19}$$

Proof. According to (5.14), quasiconvexity is defined in terms of a bounded domain. In [36] we have linked quasiconvexity to test functions in all-space. According to [36, Theorem 4.6] the existence of the radial solution $\overline{\boldsymbol{y}}(\boldsymbol{x})$ of (5.9) as in assumption (i) implies that

$$\int_{\mathbb{R}^n} \mathscr{E}(f_\infty\boldsymbol{I}_n, \nabla\overline{\boldsymbol{y}})d\boldsymbol{x} = 0,$$

as

$$\nabla\overline{\boldsymbol{y}} - f_\infty\boldsymbol{I}_n \in \mathscr{S} = \{\boldsymbol{\phi} \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n) : \nabla\boldsymbol{\phi} \in L^2(\mathbb{R}^n; \mathbb{R}^{m\times n})\},$$

according to (5.17). Then, by [36, Theorem 3.10], the assumption (ii) implies

$$\int_{\mathbb{R}^n} \mathscr{E}(f_\infty\boldsymbol{I}_n, \nabla\boldsymbol{y})d\boldsymbol{x} \geq 0$$

for all $\boldsymbol{y}(\boldsymbol{x})$, such that $\nabla\boldsymbol{y}(\boldsymbol{x}) - f_\infty\boldsymbol{I}_n \in \mathscr{S}$. In that case conclusions (a) and (b) follow from [36, Theorem 4.3].

To prove part (c) we observe that radial solutions have the property that $\overline{\boldsymbol{y}}(\boldsymbol{x}) = f_R \boldsymbol{x}$ for all $\boldsymbol{x} \in \partial B(0, R)$, where $f_R = \eta(R)/R$. Hence, by the already eastablished conclusion (b), Theorem 5.3 applies and says that

$$|B(0,R)|QW\left(\frac{\eta(R)}{R}\boldsymbol{I}_n\right) = \int_{B(0,R)} W(\nabla\overline{\boldsymbol{y}})d\boldsymbol{x}. \tag{5.20}$$

We complete the proof by the application of the Clapeyron theorem:

$$\boldsymbol{P}\boldsymbol{n} = w_1\widehat{\boldsymbol{x}}, \quad \boldsymbol{P}^* = w\boldsymbol{I}_n - \eta' w_1 \widehat{\boldsymbol{x}} \otimes \widehat{\boldsymbol{x}} + w_2\frac{\eta}{r}(\boldsymbol{I}_n - \widehat{\boldsymbol{x}} \otimes \widehat{\boldsymbol{x}})$$

Thus, according to (4.6),

$$\int_{B(0,R)} W(\nabla\overline{\boldsymbol{y}})d\boldsymbol{x} = \frac{1}{n}\int_{\partial B(0,R)} \{w_1\widehat{\boldsymbol{x}}\cdot\eta\widehat{\boldsymbol{x}} + (w - \eta' w_1)\widehat{\boldsymbol{x}}\cdot R\widehat{\boldsymbol{x}}\}\, dS.$$

We conclude that

$$QW\left(\frac{\eta(R)}{R}\boldsymbol{I}_n\right) = w_1(R)\frac{\eta(R)}{R} + w(R) - \eta'(R)w_1(R).$$

The theorem is now proved.

$\square$

5.4 Formation of a void

In this section we show how GCT can be used for solving specific problems coming from engineering applications. Specifically, we consider here a classical problem of small void nucleation in a preloaded body which plays important role in a broad range of mechanical theories [73, 66, 14, 6, 75, 80, 110, 32, 61, 2, 19, 48, 64, 79, 91, 12]. In this subsection we assume that both W and all extremals are of class C^2.

Our starting point is a classical hyperelastic body with nonlinear energy density $W(\boldsymbol{F})$ which occupies a domain Ω in its reference state and is loaded at the boundary by tractions $\boldsymbol{t}(\boldsymbol{x})$. The corresponding energy functional is

$$E = \int_\Omega W(\nabla\boldsymbol{y})d\boldsymbol{x} - \int_{\partial\Omega} \boldsymbol{t}(\boldsymbol{x})\cdot\boldsymbol{y}\, dS(\boldsymbol{x}), \tag{5.21}$$

Suppose that the applied tractions are fully equilibrated and therefore

$$\int_{\partial\Omega} \boldsymbol{t}\, dS = 0, \quad \int_{\partial\Omega} \{\boldsymbol{t}\otimes\boldsymbol{x} - \boldsymbol{x}\otimes\boldsymbol{t}\}dS = 0.$$

The resulting deformation $\boldsymbol{y}(\boldsymbol{x})$ can be found by solving the system of equations

$$\begin{cases} \nabla\cdot W_{\boldsymbol{F}}(\nabla\boldsymbol{y}) = 0, & \boldsymbol{x}\in\Omega, \\ W_{\boldsymbol{F}}(\nabla\boldsymbol{y}(\boldsymbol{x}))\boldsymbol{n} = \boldsymbol{t}(\boldsymbol{x}), & \boldsymbol{x}\in\partial\Omega. \end{cases} \tag{5.22}$$

and we assume that the solution is of class C^2.

To describe the process of void formation, we introduce $\omega \subset \mathbb{R}^n$, a bounded domain, such that $0 \in \omega$. Suppose that during the implied point ablation, at $0 \in \Omega$ a small stress-free void $\epsilon\omega$ is formed inside Ω through the removal of mass. Suppose further that $\epsilon > 0$ is so small that $\epsilon\omega \subset \Omega$, and $\Omega \setminus \epsilon\omega$ is connected.

The equilibrium deformation $\boldsymbol{y}_\epsilon$, resulting from the described mass removal, is assumed to be of class $C^2(\overline{\Omega} \setminus \epsilon\omega)$. It must then solve the system of equations

$$\begin{cases} \nabla \cdot \boldsymbol{P}(\nabla \boldsymbol{y}_\epsilon) = \boldsymbol{0}, & \boldsymbol{x} \in \Omega \setminus \epsilon\omega \\ \boldsymbol{P}(\nabla \boldsymbol{y}_\epsilon)\boldsymbol{n} = \boldsymbol{0}, & \boldsymbol{x} \in \partial(\epsilon\omega) \\ \boldsymbol{P}(\nabla \boldsymbol{y}_\epsilon)\boldsymbol{n} = \boldsymbol{t}(\boldsymbol{x}). & |\boldsymbol{x}| \in \partial\Omega \end{cases} \tag{5.23}$$

The potential energy of the resulting configuration is (up to a constant)

$$E_\epsilon = \int_{\Omega \setminus \epsilon\omega} W(\nabla \boldsymbol{y}_\epsilon) d\boldsymbol{x} - \int_{\partial\Omega} \boldsymbol{t}(\boldsymbol{x}) \cdot \boldsymbol{y}_\epsilon dS(\boldsymbol{x}), \tag{5.24}$$

We can now show how GCT can be used to obtain an expression for the energy release associated with the implied void formation and defined by the formula

$$\Delta E = \lim_{\epsilon \to 0} \frac{E_\epsilon - E}{\epsilon^n}. \tag{5.25}$$

This apparently straightforward question has caused considerable controversy in the past [20, 54, 73, 55, 96, 101, 25, 68, 53] and to the best of our knowledge has never been so far resolved in the fully nonlinear setting.

Note first that one can expect the solution $\boldsymbol{y}_\epsilon$ of the equilibrium problem (5.23) in the ablated medium to exhibit an effective boundary layer near the cavity. Such boundary layer would be different from classical boundary layers in singularly perturbed problems, where the exponential decay of the solutions within the boundary layer sets the length scale. In the case of a small cavity, it is the size ϵ of the defect that sets the length scale. In all other respects the treatment of the solution with such boundary layer can be expected to follow the standard procedure of matching the inner and outer solutions.

To shortcut the formal mathematical study and simplify the presentation, we make upfront the following assumptions which will be later at least partially justified by our explicit constructions.

(A1) The boundary value problems (5.22) and (5.23) have unique, up to an additive constant solutions $\boldsymbol{y}(\boldsymbol{x})$ and $\boldsymbol{y}_\epsilon(\boldsymbol{x})$, respectively.

(A2) There exists $\boldsymbol{w} \in C^2(\overline{\Omega} \setminus \{0\})$, such

$$\lim_{\epsilon \to 0} \frac{\boldsymbol{y}_\epsilon(\boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x})}{\epsilon^n} = \boldsymbol{w}(\boldsymbol{x}), \tag{5.26}$$

in C^1 on all compact subsets of $\overline{\Omega} \setminus \{0\}$. The assumption (5.26) is motivated by the behavior of radial solutions in the context of linear elasticity, see our Appendix A.2 for details. In particular, we also have

$$\lim_{\epsilon \to 0} \frac{\nabla \boldsymbol{y}_\epsilon(\boldsymbol{x}) - \nabla \boldsymbol{y}(\boldsymbol{x})}{\epsilon^n} = \nabla \boldsymbol{w}(\boldsymbol{x})$$

is uniform on compact subsets of $\Omega \setminus \{0\}$. The function $\boldsymbol{y}_{\rm out}(\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x}) + \epsilon^n \boldsymbol{w}(\boldsymbol{x})$ will be called the outer solution with the understanding that the normalized displacement increment $\boldsymbol{w}(\boldsymbol{x})$ governs the "far field" induced by the formation of a void.

(A3) The inner solution in the immediate vicinity of the cavity is

$$\nabla \boldsymbol{y}_\epsilon(\epsilon \boldsymbol{z}) \to \nabla \boldsymbol{y}_{\rm in}(\boldsymbol{z}), \quad \boldsymbol{z} \notin \omega \tag{5.27}$$

uniformly, on compact subsets of $\mathbb{R}^n \setminus \omega$, where $\boldsymbol{y}_{\rm in}$ is the unique solution of

$$\begin{cases} \nabla \cdot \boldsymbol{P}(\nabla \boldsymbol{y}_{\rm in}) = \boldsymbol{0}, & \boldsymbol{z} \in \mathbb{R}^n \setminus \omega \\ \boldsymbol{P}(\nabla \boldsymbol{y}_{\rm in})\boldsymbol{n} = \boldsymbol{0}, & \boldsymbol{z} \in \partial\omega \\ \nabla \boldsymbol{y}_{\rm in}(\boldsymbol{z}) \to \nabla \boldsymbol{y}(0), & |\boldsymbol{z}| \to \infty. \end{cases} \tag{5.28}$$

This assumption regarding the inner solution in (5.27) is again motivated by the behavior of radial solutions of linearly elastic isotropic exterior problem which in our case can be written in the form (see again our Appendix A.2 for details):

$$\boldsymbol{u}_{\rm in}^{\rm lin}(\boldsymbol{z}) = \frac{p\boldsymbol{z}}{n\kappa} + \frac{p\boldsymbol{z}}{2\mu(n-1)|\boldsymbol{z}|^n} = \lim_{\epsilon\to 0} \frac{\boldsymbol{u}_\epsilon(\epsilon\boldsymbol{z})}{\epsilon}, \quad |\boldsymbol{z}| > 1. \tag{5.29}$$

Since the deformation field $\boldsymbol{y}_{\rm in}(\boldsymbol{z})$ characterizes the local field in the immediate vicinity of the void, it will be interpreted as the inner solution.

(A4) There exists a constant $n \times m$ matrix $\boldsymbol{S}$, such that

$$\boldsymbol{y}_{\rm in}(\boldsymbol{z}) = \nabla \boldsymbol{y}(0)\boldsymbol{z} + \frac{\boldsymbol{S}\boldsymbol{z}}{|\boldsymbol{z}|^n} + O(|\boldsymbol{z}|^{-n}), \text{ as } |\boldsymbol{z}| \to \infty,$$

where $\boldsymbol{y}(\boldsymbol{x})$ solves (5.22).

(A5) One can replace $\nabla \boldsymbol{y}_{\rm out}(\epsilon\boldsymbol{z})$ with $\nabla \boldsymbol{y}_{\rm in}(\boldsymbol{z})$, when $|\boldsymbol{z}|$ is large, while $\epsilon|\boldsymbol{z}|$ is small.

We can now formulate our main result in the form of a theorem:

THEOREM **5.6.** *Assume that assumptions (A1)–(A5) hold. Then*

$$\Delta E = -\frac{1}{n} \int_{\partial\omega} W(\nabla \boldsymbol{y}_{\rm in}(\boldsymbol{z}))(\boldsymbol{n} \cdot \boldsymbol{z}) dS(\boldsymbol{z}). \tag{5.30}$$

Note that a related result was obtained in [91] and its relation with Theorem 5.6 is discussed in Appendix A.6.

Proof. Observe first that if we apply GCT (4.6) to the solution $\boldsymbol{y}$ of (5.22) we obtain

$$E = \frac{1}{n} \int_{\partial\Omega} \{\boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n} \cdot \boldsymbol{x}\} dS - \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{y}\, dS.$$

In view of the boundary conditions $\boldsymbol{P}\boldsymbol{n} = \boldsymbol{t}$ we also know that

$$\boldsymbol{P}^*\boldsymbol{n} = W(\nabla \boldsymbol{y})\boldsymbol{n} - (\nabla \boldsymbol{y})^T \boldsymbol{P}\boldsymbol{n} = W(\nabla \boldsymbol{y})\boldsymbol{n} - (\nabla \boldsymbol{y})^T \boldsymbol{t}.$$

Therefore

$$E = \frac{1}{n}\int_{\partial\Omega}\{\boldsymbol{t}\cdot\boldsymbol{y} + W(\nabla\boldsymbol{y})(\boldsymbol{n}\cdot\boldsymbol{x}) - \boldsymbol{t}\cdot\nabla\boldsymbol{y}\boldsymbol{x}\}dS - \int_{\partial\Omega}\boldsymbol{t}\cdot\boldsymbol{y}\,dS. \tag{5.31}$$

Similarly, since the solution $\boldsymbol{y}_\epsilon$ of (5.23) has the same boundary conditions as $\boldsymbol{y}$ on $\partial\Omega$ and traction-free boundary condition$\boldsymbol{P}(\nabla\boldsymbol{y}_\epsilon)\boldsymbol{n} = 0$ on $\partial\omega$, application of the GCT (4.6) finally gives

$$\begin{aligned}E_\epsilon = \frac{1}{n}\int_{\partial\Omega}\{\boldsymbol{t}\cdot\boldsymbol{y}_\epsilon + W(\nabla\boldsymbol{y}_\epsilon)(\boldsymbol{n}\cdot\boldsymbol{x}) - \boldsymbol{t}\cdot\nabla\boldsymbol{y}_\epsilon\boldsymbol{x}\}dS \\ - \frac{1}{n}\int_{\partial(\epsilon\omega)}W(\nabla\boldsymbol{y}_\epsilon)(\boldsymbol{n}\cdot\boldsymbol{x})dS - \int_{\partial\Omega}\boldsymbol{t}\cdot\boldsymbol{y}_\epsilon dS.\end{aligned} \tag{5.32}$$

Subtracting (5.31) and (5.32), dividing by ϵ^n and passing to the limit using our assumptions (5.26) and (5.27), we compute

$$\begin{aligned}\Delta E = \frac{1}{n}\int_{\partial\Omega}\{\boldsymbol{t}\cdot\boldsymbol{w} + \langle\boldsymbol{P}(\nabla\boldsymbol{y}), \nabla_{\partial\Omega}\boldsymbol{w}\rangle(\boldsymbol{n}\cdot\boldsymbol{x}) - \boldsymbol{t}\cdot\nabla_{\partial\Omega}\boldsymbol{w}\boldsymbol{x}\}dS \\ - \frac{1}{n}\int_{\partial\omega}\{W(\nabla\boldsymbol{y}_{\text{in}})(\boldsymbol{n}\cdot\boldsymbol{z})dS(\boldsymbol{z}) - \int_{\partial\Omega}\boldsymbol{t}\cdot\boldsymbol{w}\,dS.\end{aligned} \tag{5.33}$$

This is our preliminary formula for the desired energy increment and now we show that this formula can be simplified and expressed entirely in terms of the inner solution. Performing in (5.33) integration by parts in the terms containing $\nabla_{\partial\Omega}\boldsymbol{w}$ and using the formula

$$\int_{\partial\Omega}\boldsymbol{f}\cdot\nabla_{\partial\Omega}\phi\,dS(\boldsymbol{x}) = \int_{\partial\Omega}\{\phi(\boldsymbol{f}\cdot\boldsymbol{n})\nabla_{\partial\Omega}\cdot\boldsymbol{n} - \phi\nabla_{\partial\Omega}\cdot\boldsymbol{f}\}dS, \tag{5.34}$$

we obtain

$$\Delta E = J - \frac{1}{n}\int_{\partial\omega}W(\nabla\boldsymbol{y}_{\text{in}})(\boldsymbol{n}\cdot\boldsymbol{z})dS(\boldsymbol{z}), \tag{5.35}$$

where

$$J = \frac{1}{n}\int_{\partial\Omega}\{(\nabla_{\partial\Omega}\boldsymbol{P})[\boldsymbol{x},\boldsymbol{n},\boldsymbol{w}] - (\boldsymbol{n}\cdot\boldsymbol{x})(\nabla_{\partial\Omega}\cdot\boldsymbol{P})\cdot\boldsymbol{w}\}dS, \tag{5.36}$$

and

$$(\nabla_{\partial\Omega}\boldsymbol{P})[\boldsymbol{x},\boldsymbol{n},\boldsymbol{w}] = (\nabla_{\partial\Omega}P_i^\alpha\cdot\boldsymbol{x})n_\alpha w^i;$$

note that we assumed summation over repeated indices. To finalize the proof of the theorem we need to show that $J = 0$ which is done in our Appendix A.3. □

Remark 5.7. *We emphasize that Theorem 5.6 represents the energy release due to the formation of a traction-free void in terms of the negative of the work of configurational forces as the formula we have actually used is*

$$\Delta E = -\frac{1}{n}\int_{\partial\omega}\boldsymbol{P}^*(\nabla\boldsymbol{y}_\in)\boldsymbol{n}\cdot\boldsymbol{z}\,dS. \tag{5.37}$$

This illustrates the physical meaning of the second term in GCT (4.6), as the energy associated with creation (in this case destruction) of a body.

The derivation presented above is not the only way how GCT can be used in this problem and below we provide an alternative proof of Theorem 5.6. Here we take advantage of the known properties of the Weierstrass excess function $\mathscr{E}(\boldsymbol{F},\boldsymbol{G})$, defined in (5.1).

Alternative proof of Theorem 5.6. We begin by rewriting the work of the loading device in the problems (5.21) and (5.24) in the form of volume integral, respectively, as

$$E=\int_{\Omega}\{W(\nabla\boldsymbol{y})-\langle\boldsymbol{P}(\nabla\boldsymbol{y}),\nabla\boldsymbol{y}\rangle\}d\boldsymbol{x},$$

and

$$E_{\epsilon}=\int_{\Omega\backslash(\epsilon\omega)}\{W(\nabla\boldsymbol{y}_{\epsilon})-\langle\boldsymbol{P}(\nabla\boldsymbol{y}_{\epsilon}),\nabla\boldsymbol{y}_{\epsilon}\rangle\}d\boldsymbol{x}.$$

Rearranging the terms we can write $E_{\epsilon}-E=I_1+I_2+I_3$, where

$$I_1=\int_{\Omega\backslash(\epsilon\omega)}\mathscr{E}(\boldsymbol{F},\boldsymbol{F}_{\epsilon})d\boldsymbol{x}+\int_{\Omega\backslash(\epsilon\omega)}\langle\boldsymbol{P}(\nabla\boldsymbol{y})-\boldsymbol{P}(\nabla\boldsymbol{y}_{\epsilon}),\nabla\boldsymbol{y}_{\epsilon}-\nabla\boldsymbol{y}\rangle\}d\boldsymbol{x},$$

$$I_2=\int_{\Omega\backslash(\epsilon\omega)}\langle\boldsymbol{P}(\nabla\boldsymbol{y})-\boldsymbol{P}(\nabla\boldsymbol{y}_{\epsilon}),\nabla\boldsymbol{y}\rangle\}d\boldsymbol{x},$$

$$I_3=-\int_{\epsilon\omega}\{W(\nabla\boldsymbol{y})-\langle\boldsymbol{P}(\nabla\boldsymbol{y}),\nabla\boldsymbol{y}\rangle\}d\boldsymbol{x}.$$

Integration by parts in I_2, while taking the boundary conditions from (5.22) and (5.23) into account, gives

$$I_2=-\int_{\partial(\epsilon\omega)}\boldsymbol{P}(\nabla\boldsymbol{y})\boldsymbol{n}\cdot\boldsymbol{y}\,dS(\boldsymbol{x})=-\int_{\epsilon\omega}\langle\boldsymbol{P}(\nabla\boldsymbol{y}),\nabla\boldsymbol{y}\rangle d\boldsymbol{x},$$

where a divergence theorem on $\epsilon\omega$ was used in the second equality above. Thus,

$$I_2+I_3=-\int_{\epsilon\omega}W(\nabla\boldsymbol{y})d\boldsymbol{x}.$$

Changing variables $\boldsymbol{z}=\epsilon\boldsymbol{x}$ and passing to the limit, we obtain

$$\lim_{\epsilon\to0}\epsilon^{-n}(I_2+I_3)=-|\omega|(W(\boldsymbol{F}_0).$$

We also have

$$\epsilon^{-n}I_1=\int_{(\epsilon^{-1}\Omega\backslash\omega)}\{\mathscr{E}(\nabla\boldsymbol{y}(\epsilon\boldsymbol{z}),\nabla\boldsymbol{y}_{\epsilon}(\epsilon\boldsymbol{z}))+\langle\boldsymbol{P}(\nabla\boldsymbol{y}(\epsilon\boldsymbol{z}))-\boldsymbol{P}(\nabla\boldsymbol{y}_{\epsilon}(\epsilon\boldsymbol{z})),\nabla\boldsymbol{y}_{\epsilon}(\epsilon\boldsymbol{z})-\nabla\boldsymbol{y}(\epsilon\boldsymbol{z})\rangle\}d\boldsymbol{z},$$

To pass to the limit, we observe that $\nabla\boldsymbol{y}_{\epsilon}(\boldsymbol{x})\to\nabla\boldsymbol{y}(\boldsymbol{x})$, as $\epsilon\to0$, when $|\boldsymbol{z}|\to\infty$, such that $\boldsymbol{x}=\epsilon\boldsymbol{z}$ is finite.

We note that the main reason for using the Weierstrass excess function $\mathscr{E}(\boldsymbol{F},\boldsymbol{G})$ in this calculation is the estimate

$$|\mathscr{E}(\boldsymbol{F},\boldsymbol{G})|\le C|\boldsymbol{F}-\boldsymbol{G}|^2, \tag{5.38}$$

provided both $\boldsymbol{F}$ and $\boldsymbol{G}$ belong to a fixed compact subset of $\mathbb{R}^{m\times n}$, determining the value of the constant C in (5.38).

Note next that the second term in I_1 clearly satisfies the same estimate, and that is what allows us to replace $\nabla\boldsymbol{y}_\epsilon(\epsilon\boldsymbol{z})$ by $\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z})$, $\nabla\boldsymbol{y}(\epsilon\boldsymbol{z})$ by $\boldsymbol{F}_0=\nabla\boldsymbol{y}(0)$, and $\epsilon^{-1}\Omega$ by $\mathbb{R}^n$ in the expression for $\epsilon^{-n}I_1$. Hence,

$$\lim_{\epsilon\to 0}\epsilon^{-n}I_1=\int_{\mathbb{R}^n\setminus\omega}\{\mathscr{E}(\boldsymbol{F}_0,\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z}))+\langle\boldsymbol{P}_0-\boldsymbol{P}(\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z})),\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z})-\boldsymbol{F}_0\rangle\}d\boldsymbol{z},\tag{5.39}$$

where $\boldsymbol{F}_0=\nabla\boldsymbol{y}(0)$, and $\boldsymbol{P}_0=\boldsymbol{P}(\nabla\boldsymbol{y}(0))$.

Since $\nabla\boldsymbol{y}_{\rm in}$ is a smooth extremal of $\widehat{W}(\boldsymbol{F})=\mathscr{E}(\boldsymbol{F}_0,\boldsymbol{F})$, and because of the estimate (5.38), we can use GCT (4.6) in the first term in (5.39) and integrate by parts in the second one, while we remain in the *infinite* domain $\mathbb{R}^n\setminus\omega$. We obtain

$$\int_{\mathbb{R}^n\setminus\omega}\mathscr{E}(\boldsymbol{F}_0,\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z}))d\boldsymbol{z}=-\frac{1}{n}\int_{\partial\omega}\{(\boldsymbol{P}(\nabla\boldsymbol{y}_{\rm in})-\boldsymbol{P}_0)\boldsymbol{n}\cdot\boldsymbol{y}_{\rm in}+\mathscr{E}(\boldsymbol{F}_0,\nabla\boldsymbol{y}_{\rm in})(\boldsymbol{n},\boldsymbol{z})-$$

$$(\boldsymbol{P}(\nabla\boldsymbol{y}_{\rm in})-\boldsymbol{P}_0)\boldsymbol{n}\cdot\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z})\boldsymbol{z}\}dS=-\frac{1}{n}\int_{\partial\omega}\{\boldsymbol{P}_0\boldsymbol{n}\cdot(\nabla\boldsymbol{y}_{\rm in}\boldsymbol{z}-\boldsymbol{y}_{\rm in})+\mathscr{E}(\boldsymbol{F}_0,\nabla\boldsymbol{y}_{\rm in})(\boldsymbol{n},\boldsymbol{z})\}dS,$$

where we took into account the boundary condition $\boldsymbol{P}(\nabla\boldsymbol{y}_{\rm in})\boldsymbol{n}=0$ on $\partial\omega$. Similarly,

$$\int_{\mathbb{R}^n\setminus\omega}\langle\boldsymbol{P}_0-\boldsymbol{P}(\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z})),\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z})-\boldsymbol{F}_0\rangle d\boldsymbol{z}=-\int_{\partial\omega}\boldsymbol{P}_0\boldsymbol{n}\cdot(\boldsymbol{y}_{\rm in}(\boldsymbol{z})-\boldsymbol{F}_0\boldsymbol{z})dS=$$

$$|\omega|\langle\boldsymbol{P}_0,\boldsymbol{F}_0\rangle-\int_{\partial\omega}\boldsymbol{P}_0\boldsymbol{n}\cdot\boldsymbol{y}_{\rm in}(\boldsymbol{z})dS.$$

Expanding $\mathscr{E}(\boldsymbol{F}_0,\nabla\boldsymbol{y}_{\rm in})$, we obtain

$$-\frac{1}{n}\int_{\partial\omega}\mathscr{E}(\boldsymbol{F}_0,\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z}))(\boldsymbol{n},\boldsymbol{z})dS=-\frac{1}{n}\int_{\partial\omega}W(\nabla\boldsymbol{y}_{\rm in}(\boldsymbol{z}))(\boldsymbol{n},\boldsymbol{z})d\boldsymbol{S}+$$

$$(W(\boldsymbol{F}_0)-\langle\boldsymbol{P}_0,\boldsymbol{F}_0\rangle)|\omega|+\frac{1}{n}\int_{\partial\omega}\langle\boldsymbol{P}_0,\nabla\boldsymbol{y}_{\rm in}\rangle(\boldsymbol{n},\boldsymbol{z})dS$$

Thus,

$$\Delta E=-\frac{1}{n}\int_{\partial\omega}W(\nabla\boldsymbol{y}_{\rm in})(\boldsymbol{n}\cdot\boldsymbol{z})dS(\boldsymbol{z})+J^*,$$

where

$$J^*=-\frac{1}{n}\int_{\partial\omega}\{\boldsymbol{P}_0\boldsymbol{n}\cdot(\nabla\boldsymbol{y}_{\rm in}\boldsymbol{z}-\boldsymbol{y}_{\rm in})-\langle\boldsymbol{P}_0,\nabla\boldsymbol{y}_{\rm in}\rangle\boldsymbol{z}\cdot\boldsymbol{n}+n\boldsymbol{P}_0\boldsymbol{n}\cdot\boldsymbol{y}_{\rm in}\}\,dS.\tag{5.40}$$

To complete the proof it remains to show that $J^*=0$, which is shown by direct computation in our Appendix A.4. □

Remark 5.8. *The obtained results allow us to elucidate a delicate point, which has led to a famous "Griffith's error" in the analogous calculation conducted in the context of linear elasticity*

[40, 53]. Indeed, if instead of our way of computing the limit of $\epsilon^{-n} I_2$ above, we would have passed to the limit "directly" and replaced $\nabla \boldsymbol{y}_\epsilon(\epsilon \boldsymbol{z})$ by $\nabla \boldsymbol{y}_{\rm in}(\boldsymbol{z})$, $\nabla \boldsymbol{y}(\epsilon \boldsymbol{z})$ by $\boldsymbol{F}_0 = \nabla \boldsymbol{y}(0)$, and $\epsilon^{-1}\Omega$ by $\mathbb{R}^n$, then in the re-scaled expression $\epsilon^{-n} I_2$, we would have obtained

$$\widetilde{I}_2 = \lim_{R\to\infty} \int_{B(0,R)\setminus\omega} \langle \boldsymbol{P}_0 - \boldsymbol{P}(\nabla \boldsymbol{y}_{\rm in}), \boldsymbol{F}_0 \rangle \} d\boldsymbol{z} = -\int_{\partial\omega} \boldsymbol{P}_0 \boldsymbol{n} \cdot \boldsymbol{F}_0 \boldsymbol{z}\, dS +$$
$$\lim_{R\to\infty} \int_{\partial B(0,R)} (\boldsymbol{P}_0 - \boldsymbol{P}(\nabla \boldsymbol{y}_{\rm in}))\boldsymbol{n} \cdot \boldsymbol{F}_0 \boldsymbol{z}\, dS = -|\omega| \langle \boldsymbol{P}_0, \boldsymbol{F}_0 \rangle +$$
$$\lim_{R\to\infty} \int_{\partial B(0,R)} (\boldsymbol{P}_0 - \boldsymbol{P}(\nabla \boldsymbol{y}_{\rm in}))\boldsymbol{n} \cdot \boldsymbol{F}_0 \boldsymbol{z}\, dS,$$

which suggests that the discrepancy G between the correct answer $-|\omega|\langle \boldsymbol{P}_0, \boldsymbol{F}_0 \rangle$ and the result obtained by Griffith, who originally followed the "direct" approach presented above, was "hiding at infinity":

$$G = \lim_{R\to\infty} \int_{\partial B(0,R)} (\boldsymbol{P}_0 - \boldsymbol{P}(\nabla \boldsymbol{y}_{\rm in}))\boldsymbol{n} \cdot \boldsymbol{F}_0 \boldsymbol{z}\, dS. \tag{5.41}$$

The value of G in the special case of linear elasticity studied by Griffith in [40] is computed explicitly in Appendix A.7. Griffith himself corrected the formula for ΔE in the follow up paper [41], however, since he did not provide any details, the correct result was independently reconstructed by many other authors [99, 28, 52, 13, 94, 101, 91, 96, 17, 7].

5.5 Case study: linear elasticity

As we have already mentioned, the defining commonality of all nonlinear versions of Clapeyron's Theorem is that they express the elastic energy of a stable configuration in terms of boundary tractions, which implies both physical and configurational tractions. In this section we will juxtapose the classical Clapeyron's Theorem (1.1) against the formulas (4.6), (4.18) and (4.21) which are all relevant, given that in classical linear elasticity the energy density function is both scale-invariant and 2-homogeneous. Indeed, consider the energy density

$$W(\boldsymbol{x}, \boldsymbol{F}) = \frac{1}{2} \langle \mathsf{C}(\boldsymbol{x}) \boldsymbol{F}_{\rm sym}, \boldsymbol{F}_{\rm sym} \rangle, \tag{5.42}$$

where

$$\boldsymbol{F}_{\rm sym} = \frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}^T),$$

and where the tensor of elastic moduli $\mathsf{C}(\boldsymbol{x})$ takes values in the space of positive definite quadratic forms on $\mathrm{Sym}(\mathbb{R}^n)$—the space of $n\times n$ symmetric real matrices, and can otherwise be an arbitrary bounded and measurable function of $\boldsymbol{x}$. Note first that according to (5.42) the function $W(\boldsymbol{x}, \boldsymbol{F})$ is p-homogenous in the sense of (4.16) with $p = 2$. This means, in particular, that, given a solution of

$$\nabla \cdot (\mathsf{C}(\boldsymbol{x}) e(\boldsymbol{u})) = 0,$$

understood in the sense of distributions, we can apply formula (4.18) and obtain the classical Clapeyron theorem (1.1)

$$E[\boldsymbol{u}] := \frac{1}{2} \int_\Omega \langle \mathsf{C}(\boldsymbol{x}) e(\boldsymbol{u}), e(\boldsymbol{u}) \rangle d\boldsymbol{x} = \frac{1}{2} \int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{x}) \boldsymbol{n} \cdot \boldsymbol{u}(\boldsymbol{x}) dS(\boldsymbol{x}), \tag{5.43}$$

Note next that when the body is homogeneous and the tensor of elastic moduli C is constant, the energy density function (5.42) is both scale-free and 2-homogeneous in the sense of (4.4) and (4.16), with $p = 2$, respectively. In this case one can also show that all equilibrium solutions are smooth and therefore, by the Noether formula (2.13), they are also stationary. Hence, in addition to (4.18) that has the form (5.43) in the linearly elastic context, formulas (4.21), (4.23), and (4.6) are also applicable. However, in contrast to (5.43), these formulas involve the skew-symmetric part $w(\boldsymbol{u})$ of $\nabla \boldsymbol{u}$,

$$w(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^T),$$

representing the infinitesimal rotations. Indeed, formula (4.21) takes the form

$$\frac{1}{2}\int_{\Omega} \langle \mathsf{C}e(\boldsymbol{u}), e(\boldsymbol{u})\rangle d\boldsymbol{x} = \frac{1}{n-2}\int_{\partial\Omega} \left\{ \frac{1}{2}\langle \boldsymbol{\sigma}, e(\boldsymbol{u})\rangle \boldsymbol{n}\cdot\boldsymbol{x} - \boldsymbol{\sigma}\boldsymbol{n}\cdot(e(\boldsymbol{u}) + w(\boldsymbol{u}))\boldsymbol{x} \right\} dS(\boldsymbol{x}). \tag{5.44}$$

Formula (4.23) relating the right-hand sides of (5.43) and (5.44) can be now rewritten as an (apparently new) integral relation satisfied on the boundary by the antisymmetric part of the gradient of an equilibrium solution:

$$\int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{n}\cdot w(\boldsymbol{u})\boldsymbol{x}\, dS = \int_{\partial\Omega} \left\{ \frac{1}{2}\langle \boldsymbol{\sigma}, e(\boldsymbol{u})\rangle (\boldsymbol{n}\cdot\boldsymbol{x}) - \boldsymbol{\sigma}\boldsymbol{n}\cdot e(\boldsymbol{u})\boldsymbol{x} - \frac{n-2}{2}\boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{u} \right\} dS. \tag{5.45}$$

In the special case of $n = 2$ formula (5.45) simplifies

$$\int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{n}\cdot w(\boldsymbol{u})\boldsymbol{x}\, dS = \int_{\partial\Omega} \left\{ \frac{1}{2}\langle \boldsymbol{\sigma}, e(\boldsymbol{u})\rangle (\boldsymbol{n}\cdot\boldsymbol{x}) - \boldsymbol{\sigma}\boldsymbol{n}\cdot e(\boldsymbol{u})\boldsymbol{x} \right\} dS.$$

Finally, formula (4.6), representing the linear version of GCT, takes the form

$$\frac{1}{2}\int_{\Omega} \langle \mathsf{C}(\boldsymbol{x})e(\boldsymbol{u}), e(\boldsymbol{u})\rangle d\boldsymbol{x} = \frac{1}{n}\int_{\partial\Omega} \left\{ \boldsymbol{\sigma}\boldsymbol{n}\cdot(\boldsymbol{u} - w(\boldsymbol{u})\boldsymbol{x}) - \boldsymbol{\sigma}\boldsymbol{n}\cdot e(\boldsymbol{u})\boldsymbol{x} + \frac{1}{2}\langle \boldsymbol{\sigma}, e(\boldsymbol{u})\rangle (\boldsymbol{n}\cdot\boldsymbol{x}) \right\} dS. \tag{5.46}$$

which is clearly different from the classical Clapeyron's Theorem.

Note also that the conservation law (4.24) can be now written as a differential equation for $w(\boldsymbol{u})$ in terms of $e(\boldsymbol{u})$:

$$\nabla\cdot(\boldsymbol{\sigma}w(\boldsymbol{u})\boldsymbol{x}) = \nabla\cdot\left(\frac{1}{2}\langle \boldsymbol{\sigma}, e(\boldsymbol{u})\rangle \boldsymbol{x} - \boldsymbol{\sigma}e(\boldsymbol{u})\boldsymbol{x} - \frac{n-2}{2}\boldsymbol{\sigma}\boldsymbol{u} \right).$$

Interestingly, we can also rewrite it as a differential equation for $\nabla \boldsymbol{u}$

$$\nabla\cdot(\boldsymbol{\sigma}\nabla\boldsymbol{u}\boldsymbol{x}) = \frac{1}{2}\nabla\cdot(\langle \boldsymbol{\sigma}, e(\boldsymbol{u})\rangle \boldsymbol{x} - (n-2)\boldsymbol{\sigma}\boldsymbol{u}), \tag{5.47}$$

which in two space dimension simplifies as a representation in terms $e(\boldsymbol{u})$ only:

$$\nabla\cdot(\boldsymbol{\sigma}\nabla\boldsymbol{u}\boldsymbol{x}) = \frac{1}{2}\nabla\cdot(\langle \boldsymbol{\sigma}, e(\boldsymbol{u})\rangle \boldsymbol{x}).$$

To the best of our knowledge, formulas (5.44)–(5.47) are new. It is likely that they did not appear in the literature on linear elasticity due to the rather striking appearance in them of the skew-symmetric part of the gradient $w(\boldsymbol{u})$. Nonetheless, formulas (5.44)– (5.45) may be useful in applications, as we show, for instance, in our Appendix A.5.

Despite their fundamentally variational origin, formulas (5.44)– (5.45) could have been be also discovered by direct computation. For instance, the direct differentiation in (5.47), taking into account the equilibrium equations $\nabla \cdot \boldsymbol{\sigma} = 0$ and $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$ gives

$$\langle \boldsymbol{\sigma}, \nabla\nabla \boldsymbol{u}\boldsymbol{x} \rangle = \frac{1}{2}\nabla \langle \boldsymbol{\sigma}, e(\boldsymbol{u})\rangle \cdot \boldsymbol{x}.$$

This equation can be now verified directly using a version of Betti reciprocity which still relies essentially on the fact that the elastic tensor is constant

$$\langle \frac{\partial \boldsymbol{\sigma}}{\partial x^i}, \boldsymbol{\varepsilon}\rangle = \langle \boldsymbol{\sigma}, \frac{\partial \boldsymbol{\varepsilon}}{\partial x^i}\rangle.$$

If we turn to applications, i is instructive to first of all reconsider in the context of linear elasticity the problem of void formation which we have discussed in the general nonlinear setting in Section 5.4.

Recall that our goal in this setting is to compute the difference between the potential energies

$$E = \frac{1}{2}\int_{\Omega} \langle \mathsf{C}e(\boldsymbol{u}), e(\boldsymbol{u})\rangle d\boldsymbol{x} - \int_{\partial\Omega} \boldsymbol{t}\cdot \boldsymbol{u}\, dS,$$

and

$$E_\epsilon = \frac{1}{2}\int_{\Omega_\epsilon} \langle \mathsf{C}e(\boldsymbol{u}_\epsilon), e(\boldsymbol{u}_\epsilon)\rangle d\boldsymbol{x} - \int_{\partial\Omega} \boldsymbol{t}\cdot \boldsymbol{u}_\epsilon dS$$

of the undamaged and damaged bodies loaded by boundary tractions

$$\begin{cases} \nabla\cdot \mathsf{C}e(\boldsymbol{u}) = \boldsymbol{0}, & \boldsymbol{x}\in\Omega \\ \mathsf{C}e(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{t}(\boldsymbol{x}), & \boldsymbol{x}\in\partial\Omega \end{cases}, \qquad \begin{cases} \nabla\cdot \mathsf{C}e(\boldsymbol{u}_\epsilon) = \boldsymbol{0}, & \boldsymbol{x}\in\Omega\setminus\epsilon\omega \\ \mathsf{C}e(\boldsymbol{u}_\epsilon)\boldsymbol{n} = \boldsymbol{0}, & \boldsymbol{x}\in\partial(\epsilon\omega) \\ \mathsf{C}e(\boldsymbol{u}_\epsilon)\boldsymbol{n} = \boldsymbol{t}(\boldsymbol{x}), & \boldsymbol{x}\in\partial\Omega. \end{cases} \tag{5.48}$$

Specifically, we would like to compute

$$\Delta E = \lim_{\epsilon\to 0}\frac{E_\epsilon - E}{\epsilon^n}. \tag{5.49}$$

Instead of adapting to the case of linear elasticity our general Theorem 5.6 which relies on GCT, we use here the classical Clapeyron's Theorem (5.43) to prove an apparently different formula for ΔE:

THEOREM **5.9.**

$$\Delta E = -\frac{1}{2}\int_{\partial\omega} \boldsymbol{\sigma}(0)\boldsymbol{n}(\boldsymbol{z})\cdot \boldsymbol{u}_{\rm in}(\boldsymbol{z}) dS(\boldsymbol{z}), \tag{5.50}$$

where $\boldsymbol{u}_{\rm in}$ solves an exterior boundary value problem

$$\begin{cases} \nabla\cdot \mathsf{C}e(\boldsymbol{u}_{\rm in}) = \boldsymbol{0}, & \boldsymbol{z}\in\mathbb{R}^n\setminus\omega \\ (\mathsf{C}e(\boldsymbol{u}_{\rm in}))\boldsymbol{n} = \boldsymbol{0}, & \boldsymbol{z}\in\partial\omega \\ e(\boldsymbol{u}_{\rm in}) \to \boldsymbol{\varepsilon}_0, & |\boldsymbol{z}|\to\infty, \end{cases} \tag{5.51}$$

where $\boldsymbol{\varepsilon}_0 = e(\boldsymbol{u})(0)$, and where $\boldsymbol{\sigma}(\boldsymbol{x}) = \mathsf{C}e(\boldsymbol{u}(\boldsymbol{x}))$.

For equivalent results in different specialized settings, see [99, 28, 52, 13, 94, 101, 91, 17, 7].

Proof. We first note that by the classical Clapeyron theorem (1.1)

$$E = -\frac{1}{2}\int_{\Omega} \langle \mathsf{C}e(\boldsymbol{u}), e(\boldsymbol{u})\rangle d\boldsymbol{x}, \tag{5.52}$$

and

$$E_\epsilon = -\frac{1}{2}\int_{\Omega_\epsilon} \langle \mathsf{C}e(\boldsymbol{u}_\epsilon), e(\boldsymbol{u}_\epsilon)\rangle d\boldsymbol{x}, \tag{5.53}$$

where $\boldsymbol{u}_\epsilon$ and $\boldsymbol{u}(\boldsymbol{x})$ are the displacement vectors in the damaged and the undamaged domains, respectively.

Let $\boldsymbol{w}_\epsilon(\boldsymbol{x}) = \boldsymbol{u}_\epsilon(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{a}_\epsilon$, where $\boldsymbol{a}_\epsilon = \boldsymbol{u}_\epsilon(\epsilon\boldsymbol{z}_0) - \boldsymbol{u}(\epsilon\boldsymbol{z}_0)$, and $\boldsymbol{z}_0 \in \partial\omega$ is an arbitrarily selected, and henceforth fixed, point. Then, the vector field $\boldsymbol{w}_\epsilon$ satisfies

$$\begin{cases} \nabla\cdot \mathsf{C}e(\boldsymbol{w}_\epsilon) = \boldsymbol{0}, & \boldsymbol{x}\in\Omega\setminus\epsilon\omega \\ \mathsf{C}e(\boldsymbol{w}_\epsilon)\boldsymbol{n} = -\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}, & \boldsymbol{x}\in\partial(\epsilon\omega) \\ \mathsf{C}e(\boldsymbol{w}_\epsilon)\boldsymbol{n} = \boldsymbol{0}, & \boldsymbol{x}\in\partial\Omega. \end{cases} \tag{5.54}$$

Observe that

$$\frac{1}{2}\int_{\Omega_\epsilon} \langle \mathsf{C}e(\boldsymbol{w}_\epsilon), e(\boldsymbol{w}_\epsilon)\rangle d\boldsymbol{x} = \frac{1}{2}\int_{\Omega_\epsilon} \langle \mathsf{C}e(\boldsymbol{u}_\epsilon), e(\boldsymbol{u}_\epsilon)\rangle d\boldsymbol{x} + \frac{1}{2}\int_{\Omega_\epsilon} \langle \mathsf{C}e(\boldsymbol{u}), e(\boldsymbol{u})\rangle d\boldsymbol{x} - \int_{\Omega_\epsilon} \langle \mathsf{C}e(\boldsymbol{u}_\epsilon), e(\boldsymbol{u})\rangle d\boldsymbol{x}$$

Integrating the last term by parts, and using $\mathsf{C}e(\boldsymbol{u}_\epsilon)\boldsymbol{n} = 0$ on $\partial(\epsilon\omega)$, we obtain

$$\int_{\Omega_\epsilon} \langle \mathsf{C}e(\boldsymbol{u}_\epsilon), e(\boldsymbol{u})\rangle d\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}\cdot\boldsymbol{u}(\boldsymbol{x})dS(\boldsymbol{x}) = \int_{\Omega} \langle \mathsf{C}e(\boldsymbol{u}), e(\boldsymbol{u})\rangle d\boldsymbol{x}.$$

Therefore,

$$\frac{1}{2}\int_{\Omega_\epsilon} \langle \mathsf{C}e(\boldsymbol{w}_\epsilon), e(\boldsymbol{w}_\epsilon)\rangle d\boldsymbol{x} = E - E_\epsilon - \frac{1}{2}\int_{\epsilon\omega} \langle \mathsf{C}e(\boldsymbol{u}), e(\boldsymbol{u})\rangle d\boldsymbol{x}.$$

Then

$$E_\epsilon - E = -\frac{1}{2}\int_{\Omega_\epsilon} \langle \mathsf{C}e(\boldsymbol{w}_\epsilon), e(\boldsymbol{w}_\epsilon)\rangle d\boldsymbol{x} - \frac{1}{2}\int_{\epsilon\omega} \langle \mathsf{C}e(\boldsymbol{u}), e(\boldsymbol{u})\rangle d\boldsymbol{x}.$$

Applying Clapeyron's theorem again, this time to $\boldsymbol{w}_\epsilon$, we get

$$E_\epsilon - E = -\frac{1}{2}\int_{\partial\omega_\epsilon} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}\cdot\boldsymbol{w}_\epsilon dS(\boldsymbol{x}) - \frac{1}{2}\int_{\epsilon\omega} \langle \mathsf{C}e(\boldsymbol{u}), e(\boldsymbol{u})\rangle d\boldsymbol{x},$$

where $\boldsymbol{n}$ is the outer unit normal to $\partial\omega$. Now let us change variables $\boldsymbol{x} = \epsilon\boldsymbol{z}$ and obtain

$$E_\epsilon - E = -\frac{\epsilon^n}{2}\left(\int_{\partial\omega} \boldsymbol{\sigma}(\epsilon\boldsymbol{z})\boldsymbol{n}(\boldsymbol{z})\cdot\widetilde{\boldsymbol{u}}_\epsilon(\boldsymbol{z})dS(\boldsymbol{z}) - \int_{\omega} \langle \mathsf{C}e(\boldsymbol{u})(\epsilon\boldsymbol{z}), e(\boldsymbol{u})(\epsilon\boldsymbol{z})\rangle d\boldsymbol{z}\right),$$

where $\widetilde{\boldsymbol{u}}_\epsilon(\boldsymbol{z}) = \epsilon^{-1}\boldsymbol{w}_\epsilon(\epsilon\boldsymbol{z})$. We observe that $\nabla\widetilde{\boldsymbol{u}}_\epsilon = \nabla\boldsymbol{u}_\epsilon(\epsilon\boldsymbol{z}) - \nabla\boldsymbol{u}(\epsilon\boldsymbol{z}) \to \nabla\boldsymbol{u}_{\mathrm{in}}(\boldsymbol{z}) - \nabla\boldsymbol{u}(0)$, uniformly on compact subsets of $\mathbb{R}^n\setminus\{0\}$, while $\widetilde{\boldsymbol{u}}_\epsilon(\boldsymbol{z}_0) = 0$. It follows that $\widetilde{\boldsymbol{u}}_\epsilon(\boldsymbol{z}) \to \boldsymbol{u}_{\mathrm{in}}(\boldsymbol{z}) - \nabla\boldsymbol{u}(0)\boldsymbol{z} - \boldsymbol{a}_0$, where $\boldsymbol{a}_0 = \boldsymbol{u}_{\mathrm{in}}(\boldsymbol{z}_0) - \nabla\boldsymbol{u}(0)\boldsymbol{z}_0$. Therefore,

$$\Delta E = \lim_{\epsilon\to 0}\frac{E_\epsilon - E}{\epsilon^n} = -\frac{1}{2}\int_{\partial\omega} \boldsymbol{\sigma}(0)\boldsymbol{n}(\boldsymbol{z})\cdot(\boldsymbol{u}_{\mathrm{in}}(\boldsymbol{z}) - \nabla\boldsymbol{u}(0)\boldsymbol{z} - \boldsymbol{a}_0)dS(\boldsymbol{z}) - \frac{|\omega|}{2}\langle\boldsymbol{\sigma}(0), \boldsymbol{\varepsilon}_0\rangle.$$

Observing that

$$\int_{\partial\omega} \boldsymbol{\sigma}(0)\boldsymbol{n}(\boldsymbol{z}) \cdot \boldsymbol{a}_0 \, dS(\boldsymbol{z}) = 0,$$

and

$$\int_{\partial\omega} \boldsymbol{\sigma}(0)\boldsymbol{n}(\boldsymbol{z}) \cdot \nabla\boldsymbol{u}(0)\boldsymbol{z} \, dS(\boldsymbol{z}) = |\omega|\langle\boldsymbol{\sigma}(0), \nabla\boldsymbol{u}(0)\rangle = |\omega|\langle\boldsymbol{\sigma}(0), \boldsymbol{\varepsilon}_0\rangle,$$

we obtain

$$\Delta E = \lim_{\epsilon\to 0} \frac{E_\epsilon - E}{\epsilon^n} = -\frac{1}{2}\int_{\partial\omega} \boldsymbol{\sigma}(0)\boldsymbol{n}(\boldsymbol{z}) \cdot \boldsymbol{u}_{\rm in}(\boldsymbol{z}) \, dS(\boldsymbol{z}).$$

$\square$

The apparent difference between the conclusions of a general Theorem 5.6 and a special Theorem 5.9 is discussed in our Appendix A.5. Specifically, we show that formulas (5.30) and (5.50) are in full agreement, and that their different appearance is due to the fact that in their derivations we used two very different versions of Clapeyron's theorem. Specifically, in the proof of Theorem 5.6 we relied on the version (4.6) expressing the scale-free nature of elasticity theory. Instead, in our proof of the Theorem 5.9 we used the version (4.18) manifesting the homogeneity of degree 2 of linear elastic energy. The different forms (5.44) and (5.46) taken by the nonlinear versions of the Claperon's Theorem in the setting of linear elasticity relate the two approaches. It is also important to mention that a different path of addressing the void formation problem in the nonlinear setting was taken in [91] where the authors obtained the result more resembling (5.50), than (5.30). This is discussed vis-a-vis our own approach in Appendix A.6.

6 Conclusions

In this paper we revisited the classical Clapeyron's Theorem of the linear elasticity theory that expresses the energy of an equilibrium configuration in terms of the work of physical forces on the boundary. We derived various nonlinear analogs of this well known engineering result by revealing its link with the classical work of E. Noether on variational symmetries in nonlinear field theories. We showed that in view of this connection, the obtained family of formulas expressing "energy in terms of work" can be interpreted as a rather general statement within Calculus of Variations reflecting partial variational symmetries of the corresponding theories. Specifically, we showed that in the case of scale invariance combined with p-homogeneity one obtains not only a direct generalization of the Clapeyron's theorem of linear elasticity containing the stress tensor but also its dual version expressed through the corresponding Eshelby tensor. In this respect an important mechanical aspect of our work is that it brings together physical and configurational forces and allows one to differentiate between the part of the elastic energy stored due to the action of applied loads and the "cold work type" part of the energy emerging due to elastic incompatibility. More generally, the proposed broader reading of the Clapeyron's theorem allowed us to link it to various apparently unrelated classical results, including the Green's formula in nonlinear elastostatics and the expression of the minimal value of the equilibrium energy through the Weierstrass excess function. Of particular interest are the applications of our generalized Clapeyron Theorem (GCT) to elastodynamics where it naturally accounts for the inevitable formation of shocks. We have also shown that our study provides new useful tools allowing one to advance in various applied variational problems where explicit solutions are not readily available. In particular, we showed that GCT can be potentially useful in generating stability limits for elastic solids undergoing fracture and phase transitions.

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A Appendix

A.1 The far-field asymptotics of solutions in (5.15)

Here we prove the asymptotics (5.17) of the solutions of (5.15) by substituting the ansatz

$$\eta(r) = f_\infty r + Ar^{-\alpha} \tag{A.1}$$

into (5.15). The asymptotics at $r = \infty$ of various terms in (5.15) are as follows:

$$w_{11}\eta'' = w_{11}^\infty \frac{\alpha(\alpha+1)A}{r^{\alpha+2}} + o(r^{-(\alpha+2)}), \quad w_{12}\left(\frac{\eta}{r}\right)' = -w_{12}^\infty \frac{(\alpha+1)A}{r^{\alpha+2}} + o(r^{-(\alpha+2)}),$$

$$w_1 = w_1^\infty - w_{11}^\infty \frac{\alpha A}{r^{\alpha+1}} + (n-1)w_{12}^\infty \frac{A}{r^{\alpha+1}} + o(r^{-(\alpha+1)}),$$

$$w_2 = w_2^\infty - w_{21}^\infty \frac{\alpha A}{r^{\alpha+1}} + w_{22}^\infty \frac{A}{r^{\alpha+1}} + (n-2)w_{23}^\infty \frac{A}{r^{\alpha+1}} + o(r^{-(\alpha+1)}),$$

where

$$w_i^\infty = \frac{\partial w}{\partial v_i}(f_\infty, \dots, f_\infty), \quad w_{ij}^\infty = \frac{\partial^2 w}{\partial v_i \partial v_j}(f_\infty, \dots, f_\infty).$$

Thus, the indicial equation for α is, taking into account that

$$w_1^\infty = w_2^\infty, \quad w_{23}^\infty = w_{21}^\infty = w_{12}^\infty, \quad w_{22}^\infty = w_{11}^\infty,$$

$$w_{11}^\infty \alpha(\alpha+1) - (n-1)(\alpha+1)w_{12}^\infty + (n-1)(\alpha+1)(w_{12}^\infty - w_{11}^\infty) = 0.$$

Dividing both sides by $\alpha+1$ (since $\alpha = -1$ corresponds to the first term $f_\infty r$ in the asymptotics (A.1)) we obtain

$$w_{11}^\infty \alpha - (n-1)w_{12}^\infty + (n-1)(w_{12}^\infty - w_{11}^\infty) = 0.$$

This becomes $w_{11}^\infty(\alpha - n + 1) = 0$, and therefore, $\alpha = n-1$, as claimed.

A.2 Outer solution for a spherical void in the linear elastic media

Here we motivate the assumption that

$$\lim_{\epsilon \to 0} \frac{\boldsymbol{y}_\epsilon(\boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x})}{\epsilon^n} = \boldsymbol{w}(\boldsymbol{x}), \tag{A.2}$$

by the observed behavior of solutions in a radially symmetric isotropic linear elastic example, where

$$W_{\rm lin}(\boldsymbol{F})=\frac{\lambda}{2}(\mathrm{Tr}\,\boldsymbol{F})^2+\mu|\boldsymbol{F}_{\rm sym}|^2,$$

$\Omega=B(0,1)$ is a unit ball under uniform tension $\boldsymbol{\sigma n}=p\boldsymbol{n}$, $|\boldsymbol{x}|=1$, and $\omega=B(0,1)$ is a spherical cavity. In this case the displacements in the original unablated medium are

$$\boldsymbol{u}(\boldsymbol{x})=\frac{p\boldsymbol{x}}{n\kappa},\tag{A.3}$$

where κ is the bulk modulus of the material, while after ablation they are given by

$$\boldsymbol{u}_\epsilon^{\rm lin}(\boldsymbol{x})=\frac{p\boldsymbol{x}}{n\kappa(1-\epsilon^n)}+\frac{p\epsilon^n\boldsymbol{x}}{2\mu(n-1)(1-\epsilon^n)|\boldsymbol{x}|^n}.\tag{A.4}$$

We easily verify that

$$\boldsymbol{w}^{\rm lin}(\boldsymbol{x})=\lim_{\epsilon\to0}\frac{\boldsymbol{u}_\epsilon(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x})}{\epsilon^n}=\frac{p\boldsymbol{x}}{n\kappa}+\frac{p\boldsymbol{x}}{2\mu(n-1)|\boldsymbol{x}|^n},\quad \boldsymbol{x}\in B(0,1)\setminus\{0\},\tag{A.5}$$

where the convergence is uniform on compact subsets of $\overline{B(0,1)}\setminus\{0\}$.

A.3 Completion of the proof of Theorem 5.6

To finalize the proof of the Theorem 5.6 we show here that indeed $J=0$. We first observe that in (5.36) we can replace $\nabla_{\partial\Omega}\boldsymbol{P}$ with $\nabla\boldsymbol{P}$ and $\nabla_{\partial\Omega}\cdot\boldsymbol{P}$ with $\nabla\cdot\boldsymbol{P}=0$. Hence,

$$J=\int_{\partial\Omega}\frac{\partial P_i^\alpha(\nabla\boldsymbol{y})}{\partial x^\beta}x^\beta n_\alpha w^i dS(\boldsymbol{x}).\tag{A.6}$$

We first observe that

$$\int_{\partial\Omega}\frac{\partial P_i^\alpha(\nabla\boldsymbol{y})}{\partial x^\beta}x^\beta n_\alpha dS=\int_\Omega\frac{\partial}{\partial x_\alpha}\left(\frac{\partial P_i^\alpha(\nabla\boldsymbol{y})}{\partial x^\beta}x^\beta\right)dS=0,$$

since $\nabla\cdot\boldsymbol{P}(\nabla\boldsymbol{y}(\boldsymbol{x}))=0$ in Ω. It follows that we can replace $\boldsymbol{w}$ in the formula for J with $\boldsymbol{w}+\boldsymbol{w}_0$ for any constant $\boldsymbol{w}_0$. This constant will be chosen later as convenient. For now $\boldsymbol{w}$ will stand for $\boldsymbol{w}+\boldsymbol{w}_0$.

Now we want to apply the divergence theorem to the right-hand side of (A.6), except we need to keep in mind that $\boldsymbol{w}(\boldsymbol{x})$ has a singularity at $\boldsymbol{x}=0$. Therefore, we can apply the divergence theorem to $\Omega\setminus(B(0,M\epsilon))$, where M is a large but fixed constant. We have

$$J=\lim_{\epsilon\to0}\int_{\Omega\setminus(B(0,M\epsilon))}\frac{\partial P_i^\alpha(\nabla\boldsymbol{y})}{\partial x^\beta}\frac{\partial w^i}{\partial x^\alpha}x^\beta d\boldsymbol{x}+\int_{\partial(B(0,M\epsilon))}\frac{\partial P_i^\alpha(\nabla\boldsymbol{y})}{\partial x^\beta}x^\beta n_\alpha w^i dS(\boldsymbol{x}),$$

where in the last integral $\boldsymbol{n}(\boldsymbol{x})$ stands for the outward unit normal to $\partial(B(0,M\epsilon))$. Taking into account that

$$\frac{\partial P_i^\alpha(\nabla\boldsymbol{y})}{\partial x^\beta}=\mathsf{L}_{ij}^{\alpha\gamma}(\nabla\boldsymbol{y})\frac{\partial^2y^j}{\partial x^\gamma\partial x^\beta},\quad \mathsf{L}_{ij}^{\alpha\gamma}(\boldsymbol{F})=\frac{\partial^2W(\boldsymbol{F})}{\partial F_\alpha^i\partial F_\gamma^j},$$

we have

$$J = \lim_{\epsilon\to 0} \int_{\Omega\setminus(B(0,M\epsilon))} (\mathsf{L}(\nabla \boldsymbol{y})\nabla \boldsymbol{w})^{\gamma}_{j} \frac{\partial^2 y^j}{\partial x^\gamma \partial x^\beta} x^\beta + \int_{\partial(B(0,M\epsilon))} \mathsf{L}^{\alpha\gamma}_{ij}(\nabla \boldsymbol{y}) \frac{\partial^2 y^j}{\partial x^\gamma \partial x^\beta} x^\beta n_\alpha w^i dS(\boldsymbol{x}).$$

Observing that

$$\frac{\partial^2 y^j}{\partial x^\gamma \partial x^\beta} x^\beta = \nabla(\nabla \boldsymbol{y}(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{y}(\boldsymbol{x}) + \boldsymbol{y}(0))^j_\gamma,$$

and that $\nabla \cdot (\mathsf{L}(\nabla \boldsymbol{y})\nabla \boldsymbol{w}) = 0$, while $(\mathsf{L}(\nabla \boldsymbol{y})\nabla \boldsymbol{w})\boldsymbol{n} = 0$ on $\partial\Omega$, we obtain

$$J = \lim_{\epsilon\to 0} \int_{\partial(B(0,M\epsilon))} \mathsf{L}^{\alpha\gamma}_{ij}(\nabla \boldsymbol{y}) \left(\frac{\partial^2 y^j}{\partial x^\gamma \partial x^\beta} x^\beta n_\alpha w^i - (\frac{\partial y^j}{\partial x^\beta} x^\beta - y^j + y(0)) n_\gamma \frac{\partial w^i}{\partial x^\alpha} \right) dS. \quad \text{(A.7)}$$

We note that in order to proceed further we need to apply the matching principle, whereby $\nabla \boldsymbol{y}_\epsilon(\boldsymbol{x}) \approx \nabla \boldsymbol{y}(\boldsymbol{x}) + \epsilon^n \nabla \boldsymbol{w}(\boldsymbol{x})$, when $|\boldsymbol{x}|$ is not too small, must match $\nabla \boldsymbol{y}_\epsilon(\epsilon \boldsymbol{z}) \approx \nabla \boldsymbol{y}_{\rm in}(\boldsymbol{z})$, when $|\boldsymbol{z}|$ is not too large. Hence, $\boldsymbol{x} = \epsilon \boldsymbol{z}$, and $\boldsymbol{z} \in B(0, M)$ for some large constant M, represents the region of validity of both approximations. Thus, in order to evaluate J, we replace

$$\nabla \boldsymbol{w}(\epsilon \boldsymbol{z}) \to \epsilon^{-n}(\nabla \boldsymbol{y}_{\rm in}(\boldsymbol{z}) - \nabla \boldsymbol{y}(\epsilon \boldsymbol{z})), \quad \boldsymbol{z} \in B(0, M),$$

and

$$\boldsymbol{w}(\epsilon \boldsymbol{z}) \to \epsilon^{-n}(\epsilon \boldsymbol{y}_{\rm in}(\boldsymbol{z}) - \boldsymbol{y}(\epsilon \boldsymbol{z}) + \boldsymbol{y}(0)).$$

where the free constant in $\boldsymbol{w}$ is now chosen so that the above approximation is valid. Making the replacement in (A.7) and changing variables $\boldsymbol{x} = \epsilon \boldsymbol{z}$, we obtain

$$J = \lim_{M\to\infty} \lim_{\epsilon\to 0} \int_{\partial B(0,M)} \mathsf{L}^{\alpha\gamma}_{ij}(\nabla \boldsymbol{y}(\epsilon \boldsymbol{z})) \left[\frac{\partial^2 y^j}{\partial x^\gamma \partial x^\beta}(\epsilon \boldsymbol{z}) z^\beta n_\alpha(\boldsymbol{z})(\epsilon y^i_{\rm in}(\boldsymbol{z}) - y^i(\epsilon \boldsymbol{z}) + y^i(0)) - \right.$$
$$\left. \left(\frac{\partial y^j}{\partial x^\beta}(\epsilon \boldsymbol{z}) z^\beta - \frac{y^j(\epsilon \boldsymbol{z}) - y^j(0)}{\epsilon} \right) \left(\frac{\partial y^i_{\rm in}}{\partial z^\alpha}(\boldsymbol{z}) - \frac{\partial y^i}{\partial x^\alpha}(\epsilon \boldsymbol{z}) \right) n_\gamma(\boldsymbol{z}) \right] dS(\boldsymbol{z}) = 0,$$

because

$$\lim_{\epsilon\to 0} \left(\frac{\partial y^j}{\partial x^\beta}(\epsilon \boldsymbol{z}) z^\beta - \frac{y^j(\epsilon \boldsymbol{z}) - y^j(0)}{\epsilon} \right) = \frac{\partial y^j}{\partial x^\beta}(0) z^\beta - \frac{\partial y^j}{\partial x^\alpha}(0) z^\alpha = 0.$$

We conclude that (5.30) holds.

A.4 Completion of the alternative proof of the Theorem 5.6

To finalize the alternative proof of the Theorem 5.6 we now show that indeed $J^* = 0$, where J^* is given by (5.40). First we observe that we can write $-nJ^* = \int_{\partial\omega} \boldsymbol{v}(\boldsymbol{z}) \cdot \boldsymbol{n}(\boldsymbol{z}) dS$, where

$$v^\alpha = P^\alpha_{0i} \left(\frac{\partial y^i}{\partial z^\beta} z^\beta - y^i \right) - P^\beta_{0i} \frac{\partial y^i}{\partial z^\beta} z^\alpha + n P^\alpha_{0i} y^i.$$

Hence,

$$-nJ^* = \lim_{R\to\infty} \left[\int_{\partial(B(0,R)\setminus\omega)} \nabla \cdot \boldsymbol{v} d\boldsymbol{z} - \int_{\partial B(0,R)} \boldsymbol{v} \cdot \boldsymbol{n} \, dS \right].$$

We compute

$$\nabla\cdot\boldsymbol{v}=P^{\alpha}_{0i}\frac{\partial^2 y^i}{\partial z^\beta\partial z^\alpha}z^\beta-nP^{\beta}_{0i}\frac{\partial y^i}{\partial z^\beta}-P^{\beta}_{0i}\frac{\partial^2 y^i}{\partial z^\beta\partial z^\alpha}z^\alpha+nP^{\alpha}_{0i}\frac{\partial y^i}{\partial z^\alpha}=0.$$

Therefore,

$$nJ^*=\lim_{R\to\infty}\int_{\partial B(0,R)}\{\boldsymbol{P}_0\boldsymbol{n}\cdot(\nabla\boldsymbol{y}_{\rm in}\boldsymbol{z}-\boldsymbol{y}_{\rm in})-\langle\boldsymbol{P}_0,\nabla\boldsymbol{y}_{\rm in}\rangle(\boldsymbol{n},\boldsymbol{z})+n\boldsymbol{P}_0\boldsymbol{n}\cdot\boldsymbol{y}_{\rm in}\}\,dS.$$

We observe that we can replace $\boldsymbol{y}_{\rm in}(\boldsymbol{z})$ in the formula for nJ with $\widetilde{\boldsymbol{y}}(\boldsymbol{z})=\boldsymbol{y}_{\rm in}(\boldsymbol{z})-\boldsymbol{F}_0\boldsymbol{z}$. We also note that there exists an $n\times n$ matrix $\boldsymbol{S}$ ("polarization" tensor [1]), depending on the shape of ω, such that

$$\widetilde{\boldsymbol{y}}(\boldsymbol{z})=\frac{\boldsymbol{S}\boldsymbol{z}}{|\boldsymbol{z}|^n}+O(|\boldsymbol{z}|^{-n}),\ \text{as } |\boldsymbol{z}|\to\infty. \tag{A.8}$$

Substituting this asymptotics into the formula for nJ we obtain

$$nJ^*=\int_{\mathbb{S}^{n-1}}\{n\boldsymbol{P}_0\boldsymbol{\xi}\cdot\boldsymbol{S}\boldsymbol{\xi}-\langle\boldsymbol{P}_0,\boldsymbol{S}\rangle\}dS(\boldsymbol{\xi}). \tag{A.9}$$

It is clear that

$$\boldsymbol{M}=\int_{\mathbb{S}^{n-1}}\boldsymbol{\xi}\otimes\boldsymbol{\xi}dS(\boldsymbol{\xi})=\lambda\boldsymbol{I}_n,$$

since $\boldsymbol{R}\boldsymbol{M}\boldsymbol{R}^T=\boldsymbol{M}$ for any rotation matrix $\boldsymbol{R}$. Taking traces we obtain $n\lambda=|\mathbb{S}^{n-1}|$. It follows that

$$\int_{\mathbb{S}^{n-1}}n\boldsymbol{P}_0\boldsymbol{\xi}\cdot\boldsymbol{S}\boldsymbol{\xi}\,dS(\boldsymbol{\xi})=n\langle\boldsymbol{S}^T\boldsymbol{P}_0,\lambda\boldsymbol{I}_n\rangle=n\lambda\langle\boldsymbol{P}_0,\boldsymbol{S}\rangle=\int_{\mathbb{S}^{n-1}}\langle\boldsymbol{P}_0,\boldsymbol{S}\rangle dS.$$

Thus, we have proved that $nJ^*=0$. Hence, formula (5.30) is established by a different method, where the Clapeyron theorem has once again played a key role.

A.5 Reconciliation of Theorems 5.6 and 5.9

Here we show that the statements of a general Theorem 5.6 and a special Theorem 5.9 are in full agreement. To this end we temporarily denote the right-hand side of the nonlinear result (5.30) by $\Delta_{\rm NL}E$ and the right-hand side of the linear result (5.50) by $\Delta_L E$. Our starting point will be again the Weierstrass $\mathscr{E}$-function that ensures the proper decay at infinity for the Clapeyron theorem to operate in an exterior domain. When $W(\boldsymbol{F})$ is given y (5.42) with constant tensor of material moduli C, we compute

$$E_\infty=\int_{\mathbb{R}^n\setminus\omega}\mathscr{E}(\boldsymbol{\varepsilon}_0,\boldsymbol{\varepsilon}_{\rm in})d\boldsymbol{z}=\int_{\mathbb{R}^n\setminus\omega}\left\{\frac{1}{2}\langle\mathsf{C}\boldsymbol{\varepsilon}_{\rm in}\boldsymbol{\varepsilon}_{\rm in}\rangle-\frac{1}{2}\langle\mathsf{C}\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_0\rangle-\langle\mathsf{C}\boldsymbol{\varepsilon}_0,\boldsymbol{\varepsilon}_{\rm in}-\boldsymbol{\varepsilon}_0\rangle\right\}dS=$$

$$\frac{1}{2}\int_{\mathbb{R}^n\setminus\omega}\langle\mathsf{C}\widetilde{\boldsymbol{\varepsilon}},\widetilde{\boldsymbol{\varepsilon}}\rangle d\boldsymbol{z},$$

where $\widetilde{\varepsilon} = \varepsilon_{\rm in} - \varepsilon_0$. The energy density $\frac{1}{2}\langle \mathsf{C}\widetilde{\varepsilon}, \widetilde{\varepsilon}\rangle$ is both 2-homogeneous and scale-invariant function of $\widetilde{\boldsymbol{F}} = \nabla\widetilde{\boldsymbol{u}}$, where $\widetilde{\boldsymbol{u}} = \boldsymbol{u}_{\rm in} - \boldsymbol{F}_0\boldsymbol{z} - \boldsymbol{u}_0$, is a C^2 function, which solves

$$\begin{cases} \nabla\cdot \mathsf{C}e(\widetilde{\boldsymbol{u}}) = 0, & \boldsymbol{z}\notin\omega \\ (\mathsf{C}e(\widetilde{\boldsymbol{u}}))\boldsymbol{n} = -\boldsymbol{\sigma}_0\boldsymbol{n}, & \boldsymbol{z}\in\partial\omega, \\ \widetilde{\boldsymbol{u}}(\boldsymbol{z})\to 0, & |\boldsymbol{z}|\to\infty. \end{cases}$$

Hence, both versions (5.46) and (5.43) of Clapeyron's theorem are applicable. Moreover, our use of the Weierstrass $\mathscr{E}$-function allows us to use formulas (5.46) and (5.43) in the exterior domain. Therefore, on the one hand we have

$$E_\infty = -\frac{1}{n}\int_{\partial\omega}\left\{(\mathsf{C}\widetilde{\varepsilon})\boldsymbol{n}\cdot\widetilde{\boldsymbol{u}} + \frac{1}{2}\langle\mathsf{C}\widetilde{\varepsilon},\widetilde{\varepsilon}\rangle(\boldsymbol{z}\cdot\boldsymbol{n}) - (\mathsf{C}\widetilde{\varepsilon})\boldsymbol{n}\cdot\nabla\widetilde{\boldsymbol{u}}\boldsymbol{z}\right\}d\boldsymbol{z}, \tag{A.10}$$

due to (5.46), and on the other

$$E_\infty = -\frac{1}{2}\int_{\partial\omega}(\mathsf{C}\widetilde{\varepsilon})\boldsymbol{n}\cdot\widetilde{\boldsymbol{u}}dS = \frac{1}{2}\int_{\partial\omega}\boldsymbol{\sigma}_0\boldsymbol{n}\cdot\widetilde{\boldsymbol{u}}dS, \tag{A.11}$$

due to (5.43). The negative signs in (A.10) and (A.11) come from our convention that $\boldsymbol{n}$ denotes the *outward* unit normal to $\partial\omega$, which the the inward-pointing normal for the boundary of $\mathbb{R}^n\setminus\omega$. The similarity between formula (A.11) for E_∞ and formula (5.50) for $\Delta_L E$ is apparent. Therefore, we can rewrite (A.11) as

$$E_\infty = \Delta_L E + \frac{1}{2}\int_{\partial\omega}\boldsymbol{\sigma}_0\boldsymbol{n}\cdot(\widetilde{\boldsymbol{u}}+\boldsymbol{u}_{\rm in})dS = \Delta_L E + \int_{\partial\omega}\boldsymbol{\sigma}_0\boldsymbol{n}\cdot\widetilde{\boldsymbol{u}}dS + \frac{1}{2}\langle\boldsymbol{\sigma}_0,\varepsilon_0\rangle|\omega|. \tag{A.12}$$

Replacing $(\mathsf{C}\widetilde{\varepsilon})\boldsymbol{n}$ by $-\boldsymbol{\sigma}_0\boldsymbol{n}$ in (A.10) and expanding

$$\langle\mathsf{C}\widetilde{\varepsilon},\widetilde{\varepsilon}\rangle = \langle\mathsf{C}\varepsilon_{\rm in},\varepsilon_{\rm in}\rangle - 2\langle\mathsf{C}\varepsilon_{\rm in},\varepsilon_0\rangle + \langle\mathsf{C}\varepsilon_0,\varepsilon_0\rangle = \langle\mathsf{C}\varepsilon_{\rm in},\varepsilon_{\rm in}\rangle - 2\langle\mathsf{C}\widetilde{\varepsilon},\varepsilon_0\rangle - \langle\mathsf{C}\varepsilon_0,\varepsilon_0\rangle,$$

formula (A.10) becomes

$$E_\infty = \Delta_{NL}E - \frac{1}{n}\int_{\partial\omega}\boldsymbol{\sigma}_0\boldsymbol{n}\cdot(\nabla\widetilde{\boldsymbol{u}}\boldsymbol{z}-\widetilde{\boldsymbol{u}}) - \langle\boldsymbol{\sigma}_0,\widetilde{\varepsilon}\rangle(\boldsymbol{z}\cdot\boldsymbol{n})dS + \frac{1}{2}\langle\boldsymbol{\sigma}_0,\varepsilon_0\rangle|\omega|. \tag{A.13}$$

Hence, formulas (A.12) and (A.13) show that $\Delta_{NL}E = \Delta_L E$ if and only if $J=0$, where

$$J = \int_{\partial\omega}\{\boldsymbol{\sigma}_0\boldsymbol{n}\cdot(\nabla\widetilde{\boldsymbol{u}}\boldsymbol{z}-\widetilde{\boldsymbol{u}}) - \langle\boldsymbol{\sigma}_0,\widetilde{\varepsilon}\rangle(\boldsymbol{z}\cdot\boldsymbol{n}) + n\boldsymbol{\sigma}_0\boldsymbol{n}\cdot\widetilde{\boldsymbol{u}}\}dS.$$

As in the analysis in Appendix A.4 we first write write $J = \int_{\partial\omega}\boldsymbol{v}\cdot\boldsymbol{n}\,dS$ and then verify that $\nabla\cdot\boldsymbol{v}=0$. Indeed,

$$v^\alpha = \sigma_{0i}^\alpha\left(\frac{\partial u^i}{\partial z^\beta}z^\beta - u^i\right) - \sigma_{0i}^\beta\frac{\partial u^i}{\partial z^\beta}z^\alpha + n\sigma_{0i}^\alpha u^i.$$

Hence,

$$\nabla\cdot\boldsymbol{v} = \sigma_{0i}^\alpha\left(\frac{\partial^2 u^i}{\partial z^\alpha\partial z^\beta}z^\beta + \frac{\partial u^i}{\partial z^\beta}\delta_\alpha^\beta - \frac{\partial u^i}{\partial z^\alpha}\right) - \sigma_{0i}^\beta\frac{\partial^2 u^i}{\partial z^\alpha\partial z^\beta}z^\alpha - n\sigma_{0i}^\beta\frac{\partial u^i}{\partial z^\beta} + n\sigma_{0i}^\alpha\frac{\partial u^i}{\partial z^\alpha} = 0.$$

It follows that

$$J = -\lim_{R\to\infty} \int_{\partial B_R} \{\boldsymbol{\sigma}_0 \boldsymbol{n} \cdot (\nabla \widetilde{\boldsymbol{u}} \boldsymbol{z} - \widetilde{\boldsymbol{u}}) - \langle \boldsymbol{\sigma}_0, \widetilde{\boldsymbol{\varepsilon}} \rangle (\boldsymbol{z} \cdot \boldsymbol{n}) + n \boldsymbol{\sigma}_0 \boldsymbol{n} \cdot \widetilde{\boldsymbol{u}}\} dS.$$

As in Appendix A.4 we use (A.8) and the calculation fully analogous to (A.9) shows that $J = 0$, establishing the equality of $\Delta_{NL}E$ and $\Delta_L E$.

A.6 Rice and Drucker's analysis of void formation

In this section we revisit, for the sake of completeness, the calculation of the total energy release upon the removal of a subregion $\omega \subset \Omega$, due to Rice and Drucker [91] because their result is in some sense intermediate between the asymptotic formulas (5.30) and (5.50) and because their argument is conducted almost entirely in the context of nonlinear elasticity, while leading to a formula resembling (5.50).

Consider a body Ω loaded by a combination of "fixed grips" (displacement boundary conditions) on $\partial\Omega_D$, tractions on $\partial\Omega_N$, and body forces in Ω. The total energy functional is[4]

$$E[\boldsymbol{y}] = \int_\Omega \{W(\boldsymbol{x}, \nabla \boldsymbol{y}) - \boldsymbol{b} \cdot \boldsymbol{y}\} d\boldsymbol{x} - \int_{\partial\Omega_N} \boldsymbol{y} \cdot \boldsymbol{t} \, dS,$$

where for notational convenience we used $\boldsymbol{y}(\boldsymbol{x})$ instead of the displacements $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{x}$ in the work of external body forces and boundary tractions. We emphasize that the explicit dependence of the energy density W on $\boldsymbol{x}$ permits heterogeneous materials and location-dependent prestress.

Let $\boldsymbol{y}_0$ be the energy minimizer solving

$$\begin{cases} \nabla \cdot \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_0) = \boldsymbol{b}, & \boldsymbol{x} \in \Omega, \\ \boldsymbol{y}_0(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega_D, \\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_0)\boldsymbol{n} = \boldsymbol{t}, & \boldsymbol{x} \in \partial\Omega_N, \end{cases} \tag{A.14}$$

where $\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{x}$ are the prescribed displacments of the Dirichlet part $\partial\Omega_D$ of the boundary. Let $\boldsymbol{y}_d$ be the energy minimizer of

$$E_d[\boldsymbol{y}] = \int_{\Omega\setminus\omega} \{W(\boldsymbol{x}, \nabla \boldsymbol{y}) - \boldsymbol{b} \cdot \boldsymbol{y}\} d\boldsymbol{x} - \int_{\partial\Omega_N} \boldsymbol{y} \cdot \boldsymbol{t} \, dS,$$

where $\omega \subset \Omega$ is the damaged region that has been destroyed. The deformation $\boldsymbol{y}_d$ solves the boundary value problem

$$\begin{cases} \nabla \cdot \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_d) = \boldsymbol{b}, & \boldsymbol{x} \in \Omega \setminus \omega, \\ \boldsymbol{y}_d(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega_D, \\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_d)\boldsymbol{n} = \boldsymbol{t}, & \boldsymbol{x} \in \partial\Omega_N, \\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_d)\boldsymbol{n} = 0, & \boldsymbol{x} \in \partial\omega. \end{cases} \tag{A.15}$$

[4] In [91] Rice and Drucker were working in the context of geometrically linear, physically nonlinear elasticity. However, nothing in their analysis, reproduced below appealed to the assumption of geometric linearity.

Our goal is to find the formula for the energy increment

$$\Delta E = E[\boldsymbol{y}_0] - E_d[\boldsymbol{y}_d]$$

expressed entirely in terms of the fields on $\partial\omega$.

We begin with the obvious expression

$$E[\boldsymbol{y}_0] - E[\boldsymbol{y}_d] = E_\omega + \int_{\Omega\setminus\omega}\{W(\boldsymbol{x},\boldsymbol{F}_0) - W(\boldsymbol{x},\boldsymbol{F}_d) - \boldsymbol{b}\cdot(\boldsymbol{y}_0 - \boldsymbol{y}_d)\}d\boldsymbol{x} - \int_{\partial\Omega_N}\boldsymbol{t}\cdot(\boldsymbol{y}_0 - \boldsymbol{y}_d)dS,$$

where we used the shorthand $\boldsymbol{F}_0 = \nabla\boldsymbol{y}_0$ and $\boldsymbol{F}_d = \nabla\boldsymbol{y}_d$, and where

$$E_\omega = \int_\omega\{W(\boldsymbol{x},\boldsymbol{F}_0) - \boldsymbol{b}\cdot\boldsymbol{y}_0\}d\boldsymbol{x}.$$

In what follows we use the shorthand

$$\Delta' E = E[\boldsymbol{y}_0] - E[\boldsymbol{y}_d] - E_\omega$$

in order not to write E_ω in every formula. The first observation of Rice and Drucker is that for any smooth test field $\boldsymbol{u}(\boldsymbol{x})$ vanishing on $\partial\Omega_D$, we have, via integration by parts and equilibrium equations (A.15),

$$\int_{\Omega\setminus\omega}\langle\boldsymbol{P}(\boldsymbol{x},\boldsymbol{F}_d),\nabla\boldsymbol{u}\rangle d\boldsymbol{x} = \int_{\Omega\setminus\omega}\boldsymbol{b}\cdot\boldsymbol{u}d\boldsymbol{x} + \int_{\partial\Omega_N}\boldsymbol{t}\cdot\boldsymbol{u}\,dS. \tag{A.16}$$

We then use formula (A.16) with $\boldsymbol{u} = \boldsymbol{y}_0 - \boldsymbol{y}_d$ and obtain

$$\Delta' E = \int_{\Omega\setminus\omega}\{W(\boldsymbol{x},\boldsymbol{F}_0) - W(\boldsymbol{x},\boldsymbol{F}_d) - \langle\boldsymbol{P}(\boldsymbol{x},\boldsymbol{F}_d),\boldsymbol{F}_0 - \boldsymbol{F}_d\rangle\}d\boldsymbol{x} = \int_{\Omega\setminus\omega}\mathscr{E}(\boldsymbol{F}_d,\boldsymbol{F}_0)d\boldsymbol{x},$$

where the Weierstrass $\mathscr{E}$-function is defined in (5.1).

Next, Rice and Drucker use the following construction. Let $\boldsymbol{t}_\omega(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x},\boldsymbol{F}_0(\boldsymbol{x}))\boldsymbol{n}_{\partial\omega}(\boldsymbol{x})$, $\boldsymbol{x}\in\partial\omega$, where $\boldsymbol{n}_{\partial\omega}$ denotes the *inward* unit normal to $\partial\omega$ (i.e., the outward unit normal for $\partial(\Omega\setminus\omega)$). For any $s\in[0,1]$ let $\boldsymbol{y}_s(\boldsymbol{x})$ be the the *equilibrium* in $\Omega\setminus\omega$ with the same boundary conditions on $\partial\Omega$, the same body force loading on $\Omega\setminus\omega$ as $\boldsymbol{y}_0$, and satisfying $\boldsymbol{P}(\boldsymbol{x},\boldsymbol{F}_s)\boldsymbol{n}_{\partial\omega} = \boldsymbol{t}_s(\boldsymbol{x})$, where $\boldsymbol{t}_s$ depends smoothly on s and is such that $\boldsymbol{t}_0 = \boldsymbol{t}_\omega$, while $\boldsymbol{t}_1 = 0$. For example, $\boldsymbol{t}_s = (1-s)\boldsymbol{t}_\omega$ is the a natural choice. We note that $\boldsymbol{y}_s = \boldsymbol{y}_0$, when $s=0$, and $\boldsymbol{y}_1 = \boldsymbol{y}_d$. Next Rice and Drucker use the observations:

$$W(\boldsymbol{x},\boldsymbol{F}_0) - W(\boldsymbol{x},\boldsymbol{F}_d) = -\int_0^1\langle\boldsymbol{P}(\boldsymbol{x},\boldsymbol{F}_s),\nabla\dot{\boldsymbol{y}}_s\rangle ds, \quad \boldsymbol{F}_0 - \boldsymbol{F}_d = -\int_0^1\nabla\dot{\boldsymbol{y}}_s ds,$$

where $\dot{\boldsymbol{y}}_s = \dfrac{\partial\boldsymbol{y}_s}{\partial s}$, so that

$$\Delta' E = \int_{\Omega\setminus\omega}\int_0^1\{\langle\boldsymbol{P}(\boldsymbol{x},\boldsymbol{F}_d) - \boldsymbol{P}(\boldsymbol{x},\boldsymbol{F}_s),\nabla\dot{\boldsymbol{y}}_s\rangle ds d\boldsymbol{x}.$$

Now, recalling that $\boldsymbol{y}_s$ is an equilibrium configuration for each $s \in [0,1]$, switching the order of integration, and integrating by parts we obtain

$$\Delta' E = \int_0^1 \int_{\partial\omega} (\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_d)\boldsymbol{n}_{\partial\omega} - \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_s)\boldsymbol{n}_{\partial\omega}) \cdot \dot{\boldsymbol{y}}_s dS ds$$

Recalling that $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_d)\boldsymbol{n}_{\partial\omega} = 0$ we obtain formula (12) from [91] for the the energy increment in the fully nonlinear and heterogeneous setting.

$$\Delta E = \int_\omega \{W(\boldsymbol{x}, \boldsymbol{F}_0) - \boldsymbol{b} \cdot \boldsymbol{y}_0\} d\boldsymbol{x} - \int_{\partial\omega} \int_0^1 \boldsymbol{t}_s \cdot \dot{\boldsymbol{y}}_s ds dS. \tag{A.17}$$

Integrating by parts in the integral in s, and using the formula

$$\int_{\partial\omega} \boldsymbol{t}_\omega \cdot \boldsymbol{y}_0 = \int_\omega \{\boldsymbol{b} \cdot \boldsymbol{y}_0 - \langle \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_0), \boldsymbol{F}_0 \rangle\} d\boldsymbol{x},$$

we obtain our final result—a simplification of Rice-Drucker formula (A.17)

$$\Delta E = \int_\omega \{W(\boldsymbol{x}, \boldsymbol{F}_0) - \langle \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_0), \boldsymbol{F}_0 \rangle\} d\boldsymbol{x} - \int_{\partial\omega} \int_0^1 \boldsymbol{t}_\omega \cdot \boldsymbol{y}_s ds dS, \tag{A.18}$$

when $\boldsymbol{t}_s = (1-s)\boldsymbol{t}_\omega$.

If one does not look too closely, formula (A.17) resembles more (5.50) for linear elasticity than the fully nonlinear fromula (5.30). Taking a more detailed look, formula (A.18) involves a virtually intractable object $\boldsymbol{y}_s[\boldsymbol{t}_s]$, solving

$$\begin{cases} \nabla \cdot \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_s) = \boldsymbol{b}, & \boldsymbol{x} \in \Omega \setminus \omega, \\ \boldsymbol{y}_s(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega_D, \\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_s)\boldsymbol{n} = \boldsymbol{t}, & \boldsymbol{x} \in \partial\Omega_N, \\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{F}_s)\boldsymbol{n} = \boldsymbol{t}_s, & \boldsymbol{x} \in \partial\omega. \end{cases} \tag{A.19}$$

This prevents any further simplification of (A.18) in the general case.

In the case of linear elasticity, equations (A.19) become linear, and if we set $\boldsymbol{t}_s = (1-s)\boldsymbol{t}_\omega$, then, by linearity of equations (A.19), $\boldsymbol{y}_s = (1-s)\boldsymbol{y}_0 + s\boldsymbol{y}_d$. In that case equation (A.17) gives

$$\Delta E = E_\omega - \int_{\partial\omega} \int_0^1 (1-s)\boldsymbol{t}_\omega \cdot (\boldsymbol{y}_d - \boldsymbol{y}_0) ds dS = E_\omega - \frac{1}{2} \int_{\partial\omega} \boldsymbol{t}_\omega \cdot (\boldsymbol{y}_d - \boldsymbol{y}_0) dS.$$

In the presence of arbitrary inhomogeneities, but in the absence of prestress (at least in ω), the original Clapeyron theorem (1.1) applies to the solution $\boldsymbol{y}_0$ in ω and we have

$$E_\omega = -\frac{1}{2} \int_{\partial\omega} \boldsymbol{t}_\omega \cdot \boldsymbol{y}_0,$$

where the minus sign is due to the defition of $\boldsymbol{t}_\omega$ in terms of the inward unit normal to $\partial\omega$. In this case, we obtain the final formula for linear elasticity with no prestress in ω,

$$E[\boldsymbol{y}_0] - E[\boldsymbol{y}_d] = -\frac{1}{2} \int_{\partial\omega} \boldsymbol{t}_\omega \cdot \boldsymbol{y}_d dS. \tag{A.20}$$

This formula is virtually identical to (5.50), and in fact easily reduces to it in the volume-rescaled limit, when ω is replaced with $\epsilon\omega$ and when the body force vanishes at 0. In fact, the Rice-Drucker analysis presented here shows that the asymptotic formula (5.50) is valid in far greater generality than in Section 5.5.

In the case of nonlinear elasticity, when the material is homogeneous, body forces are absent and ω is replaced by $\epsilon\omega$, formula (A.18) implies

$$\lim_{\epsilon\to 0}\frac{E[\boldsymbol{y}_0]-E[\boldsymbol{y}_\epsilon]}{\epsilon^n}=|\omega|\{W_0-\langle\boldsymbol{P}_0,\boldsymbol{F}_0(0)\rangle\}-\int_{\partial\omega}\int_0^1\boldsymbol{P}_0\boldsymbol{n}_{\partial\omega}(\boldsymbol{z})\cdot\boldsymbol{y}_s^{\rm in}(\boldsymbol{z})dsdS(\boldsymbol{z}),\quad\text{(A.21)}$$

where $W_0=W(\boldsymbol{F}_0(0))$, and $\boldsymbol{y}_s^{\rm in}(\boldsymbol{z})$ solves the exterior problem

$$\begin{cases}\nabla\cdot\boldsymbol{P}(\nabla\boldsymbol{y}_s^{\rm in}(\boldsymbol{z}))=0, & \boldsymbol{z}\in\mathbb{R}^n\setminus\omega,\\ \boldsymbol{P}(\nabla\boldsymbol{y}_s^{\rm in}(\boldsymbol{z}))\boldsymbol{n}_{\partial\omega}=(1-s)\boldsymbol{P}(\boldsymbol{F}_0(0))\boldsymbol{n}_{\partial\omega}, & \boldsymbol{z}\in\partial\omega,\\ \nabla\boldsymbol{y}_s^{\rm in}(\boldsymbol{z})\to\boldsymbol{F}_0(0), & \text{as } |\boldsymbol{z}|\to\infty.\end{cases}\quad\text{(A.22)}$$

We observe that Theorem 5.6 gives another formula for ΔE above and therefore we obtain the following theorem.

THEOREM **A.1.** *Suppose $\boldsymbol{y}_s^{\rm in}$, $s\in[0,1]$ solves the exterior boundary value problem (A.22), while $\boldsymbol{y}_0$ solves the boundary value problem (5.22). Then*

$$\int_{\partial\omega}\boldsymbol{P}_0\boldsymbol{n}\cdot\int_0^1\boldsymbol{y}_s^{\rm in}dsdS(\boldsymbol{z})=\frac{1}{n}\int_{\partial\omega}W(\nabla\boldsymbol{y}_1^{\rm in})(\boldsymbol{n}\cdot\boldsymbol{z})dS(\boldsymbol{z})-|\omega|\{W_0-\langle\boldsymbol{P}_0,\boldsymbol{F}_0(0)\rangle\},$$

where $\boldsymbol{n}$ is the outward *unit normal to $\partial\omega$, which we can also rewrite as*

$$\int_{\partial\omega}\boldsymbol{P}_0\boldsymbol{n}\cdot\int_0^1(\boldsymbol{y}_s^{\rm in}(\boldsymbol{z})-\boldsymbol{F}_0(0)\boldsymbol{z})dsdS(\boldsymbol{z})=\frac{1}{n}\int_{\partial\omega}(W(\nabla\boldsymbol{y}_1^{\rm in})-W_0)(\boldsymbol{n}\cdot\boldsymbol{z})dS(\boldsymbol{z}).\quad\text{(A.23)}$$

The physical meaning of formula (A.23), i.e. of equivalence of formulas (5.30) and (A.18) is that the energy increment ΔE can be represented as a portion of the work of configurational forces that "create" a void.

A.7 Computation of the Griffith's discrepancy term (5.41)

In order to compute the limit in (5.41) we need to use the far-field asymptotics (A.8), obtaining

$$(\boldsymbol{P}(\nabla\boldsymbol{y}_{\rm in})-\boldsymbol{P}_0)\boldsymbol{n}\cdot\boldsymbol{F}_0\boldsymbol{z}=\frac{(\mathsf{C}\boldsymbol{S})\boldsymbol{n}\cdot\boldsymbol{F}_0\boldsymbol{z}}{|\boldsymbol{z}|^n}-n\frac{(\mathsf{C}(\boldsymbol{S}\boldsymbol{z}\otimes\boldsymbol{z}))\boldsymbol{n}\cdot\boldsymbol{F}_0\boldsymbol{z}}{|\boldsymbol{z}|^{n+2}}+O(|\boldsymbol{z}|^{-n}),$$

where $\mathsf{C}=W_{\boldsymbol{F}\boldsymbol{F}}(\boldsymbol{F}_0)$, and $\boldsymbol{S}$ is the polarization tensor [1] that depends in the shape of ω. Using the fact that $|\boldsymbol{z}|=R$ and $\boldsymbol{n}=\widehat{\boldsymbol{z}}=\boldsymbol{z}/|\boldsymbol{z}|$ on $\partial B(0,R)$, we obtain

$$G=\int_{\mathbb{S}^{n-1}}\{n(\mathsf{C}(\boldsymbol{S}\widehat{\boldsymbol{z}}\otimes\widehat{\boldsymbol{z}}))\widehat{\boldsymbol{z}}\cdot\boldsymbol{F}_0\widehat{\boldsymbol{z}}-(\mathsf{C}\boldsymbol{S})\widehat{\boldsymbol{z}}\cdot\boldsymbol{F}_0\widehat{\boldsymbol{z}}\}\,dS(\widehat{\boldsymbol{z}}).$$

Thus, taking into account that

$$\int_{\mathbb{S}^{n-1}} z^i z^j dS = \frac{|\mathbb{S}^{n-1}|}{n}\delta^{ij}, \qquad \int_{\mathbb{S}^{n-1}} z^i z^j z^k z^l dS = \frac{|\mathbb{S}^{n-1}|}{n(n+2)}\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right),$$

we obtain

$$G = \frac{|B(0,1)|}{n+2}(C_{ijkl}(nS_{kj}F_{il} - 2S_{kl}F_{ij}) + nC_{ijkj}S_{kl}F_{il}), \tag{A.24}$$

where the summation over repeated indices is assumed, and where we used the relation $|\mathbb{S}^{n-1}| = n|B(0,1)|$ between the surface and the volume of the unit ball.

If C is an isotropic elastic tensor with Lamé moduli λ and μ, then

$$\mathsf{C}(\boldsymbol{S}\widehat{\boldsymbol{z}} \otimes \widehat{\boldsymbol{z}})\widehat{\boldsymbol{z}} \cdot \boldsymbol{F}_0\widehat{\boldsymbol{z}} = (\lambda + \mu)(\boldsymbol{S}\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{z}})(\boldsymbol{F}_0\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{z}}) + \mu(\boldsymbol{S}\widehat{\boldsymbol{z}}, \boldsymbol{F}_0\widehat{\boldsymbol{z}}),$$

$$(\mathsf{C}\boldsymbol{S})\widehat{\boldsymbol{z}} \cdot \boldsymbol{F}_0\widehat{\boldsymbol{z}} = \lambda(\boldsymbol{F}_0\widehat{\boldsymbol{z}} \cdot \widehat{\boldsymbol{z}})\mathrm{Tr}\,\boldsymbol{S} + 2\mu\boldsymbol{S}\widehat{\boldsymbol{z}} \cdot \boldsymbol{F}_0\widehat{\boldsymbol{z}},$$

and thus,

$$G = \frac{|B(0,1)|}{n+2}((n\mu - 2\lambda)\mathrm{Tr}\,\boldsymbol{F}_0\mathrm{Tr}\,\boldsymbol{S} + (2\lambda n + \mu(n^2 + 2n - 4))\langle \boldsymbol{S}, \boldsymbol{F}_0\rangle). \tag{A.25}$$

When $n = 2$ we have

$$G = \frac{\pi}{2}((\mu - \lambda)\mathrm{Tr}\,\boldsymbol{F}_0\mathrm{Tr}\,\boldsymbol{S} + 2(\lambda + \mu)\langle \boldsymbol{S}, \boldsymbol{F}_0\rangle),$$

When $n = 3$ we have

$$G = \frac{4\pi}{15}((3\mu - 2\lambda)\mathrm{Tr}\,\boldsymbol{F}_0\mathrm{Tr}\,\boldsymbol{S} + (6\lambda + 11\mu)\langle \boldsymbol{S}, \boldsymbol{F}_0\rangle).$$

If $\boldsymbol{F}_0 = \frac{p}{n\kappa}\boldsymbol{I}_n$, then

$$G = \frac{2(n-1)p\mu|B(0,1)|}{n\kappa}\mathrm{Tr}\,\boldsymbol{S} = \begin{cases} \frac{\pi\mu p}{\kappa}\mathrm{Tr}\,\boldsymbol{S}, & n = 2, \\ \frac{16\pi\mu p}{9\kappa}\mathrm{Tr}\,\boldsymbol{S}, & n = 3. \end{cases}$$

The case of nucleating a small spherical cavity in a body occupied by isotropic linearly elastic material is exactly solvable. The solution of (5.51) with $\boldsymbol{F}_0 = \frac{p}{n\kappa}\boldsymbol{I}_n$ is

$$\boldsymbol{y}_{\mathrm{in}}(\boldsymbol{z}) = \left(\frac{p}{n\kappa} + \frac{p|\boldsymbol{z}|^{-n}}{2(n-1)\mu}\right)\boldsymbol{z},$$

resulting in $\boldsymbol{S} = \frac{p}{2(n-1)\mu}\boldsymbol{I}_n$. Thus,

$$G = \frac{p^2|B(0,1)|}{\kappa} = \begin{cases} \frac{\pi p^2}{\kappa}, & n = 2, \\ \frac{4\pi p^2}{3\kappa}, & n = 3. \end{cases}$$

In this case all of our results in this section can also be verified by a direct calculation, using the explicit solutions (A.3), (A.4) of (5.48). Needless to say, the obtained results specialized to the appropriate value of n and using the relevant polarization tensor $\boldsymbol{S}$, agree with the correct formula presented by Griffith in his second paper [41].

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