A generalization of the Chandler Davis convexity theorem.

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Abstract

In 1957 Chandler Davis proved a theorem that a rotationally invariant function on symmetric matrices is convex if and only if it is convex on the diagonal matrices. We generalize this result to groups acting nonlinearly on convex subsets of arbitrary vector spaces thereby understanding the abstract mechanism behind the classical theorem. We apply the new theorem to a problem from the mathematical theory of composite materials and derive its corollaries in the Lie algebra setting. Using the latter, we show that the Pfaffian is log-concave.

1 Introduction

The classical theorem of Chandler Davis [4] says that a rotationally invariant function on symmetric matrices is convex if and only if it is convex on the diagonal matrices. In this paper we uncover a simple abstract mechanism behind this classical theorem. A motivation for the present work comes from applied mathematics. In a problem from the mathematical theory of composite materials it was important to understand how to construct smallest convex and rotationally invariant sets containing a given set [17, Chapter 30]. The new twist was that the action of the rotation group and convexity happen in *different* coordinate systems. Equivalently, we may consider a problem in a single coordinate system but with a *non-linear* group action. The combination of convexity and rotational invariance drew us to Davis's theorem, since it seemed to be the only result of that nature. (We are aware of Atiyah-Guillemin-Sternberg theorem [1, 10], but we do not know if there is any relation of their result to ours. In particular we do not know if there is a symplectic structure hiding somewhere in our general construction of Section 2, or at least in the construction of Section 3.) Our composite materials example, discussed in detail in Section 3, serves two purposes. One is to illustrate our main theorem (Theorem 1 below). The other is to attract attention of a wider mathematical audience to a problem that is of central importance in the mathematical theory of composite materials and at the same time may have a wider appeal.

In this paper we generalize Davis's theorem to a broader context that includes, most notably, arbitrary groups, non-linear group actions and infinite dimensional spaces. In Davis's original proof it was important that the rotations act linearly on matrices. Our approach is more general and completely different from the Davis's original proof or from other proofs that have appeared up to now. Our results are new even for the linear group action.

All prior generalizations of the Davis's convexity theorem are for finite dimensional representations of Lie groups. They follow from our Theorems 1 and 2 often by way of Theorems 7–9 below. Theorems 7–9 can be thought of as generalizations of the von Neumann theorem [20] that among symmetric matrices $\boldsymbol{A}$ with the same set of eigenvalues the one that minimizes $\mathrm{Tr}\,(\boldsymbol{AB})$ should be simultaneously diagonalizable with $\boldsymbol{B}$. Lewis [12] invented his "normal decomposition systems" in order to abstract from von Neumann's result what was essential for the proof of Davis's convexity theorem. Our Theorems 1 and 2 are a next step in the abstraction process. In particular our condition (2) below is always satisfied in the framework of Lewis's "normal decomposition systems" for linear group actions.

A Lie-algebraic generalization of Davis's convexity theorem was first obtained by Lewis [13], who used Kostant's convexity theorem [11] in order to prove that there is a normal decomposition system. Similarly, Kostant's convexity theorem can be used to show that our condition (2) is satisfied, as we show after the proof of Theorem 2. We realized, however, that we do not really need the full strength of Kostant's convexity theorem in order to achieve our goal. In its place we can use much more general statements from Theorems 7–9. We therefore, extend and simplify Lewis's theorem without appealing to Kostant's result. We use our technique to prove that the Pfaffian is log-concave.

2 Generalized convexity theorem of Chandler Davis

Let C be a convex subset of a vector space V (which can be infinite dimensional). Let G be a group acting on C. For each $x \in C$ we denote $\mathcal{O}_x = \{gx \mid g \in G\}$ the orbit of x under the group action.

Definition 1 *A convex subset T of C is called a transversal if T meets every orbit*

$$\forall x \in C \ \mathcal{O}_x^T \equiv T \cap \mathcal{O}_x \neq \emptyset.$$

Notice that our definition of a transversal differs from the usual ones. For notational convenience we also define

$$G_x^T = \{g \in G : \ gx \in T\}.$$

From the definition of the transversal we get that $G_x^T \neq \emptyset$ for any $x \in C$.

Definition 2 *A function $f : C \to \mathbb{R}$ is called G-invariant if*

$$\forall x \in C \ \forall g \in G \ f(gx) = f(x).$$

We denote V' the space of linear functionals on V (we cannot talk about continuous linear functionals because we did not introduce any topology on V). Let T^* be any subset of V'. We define a set of "(T,T^*)-regular" convex functions on T by analogy with the classical theory of convex duality on locally convex topological vector spaces.

Definition 3 *Let*

$$\Gamma(T,T^*) = \{f(x) = \sup_{y\in T^*}(y(x) - \alpha(y)) \mid \alpha : T^* \to \mathbb{R}\cup\{+\infty\}\}$$

be the set of convex functions on T that are suprema of affine functions with slopes in T^*. Both functions f and α are allowed to take on the value $+\infty$. The only restriction is that α cannot be identically equal to $+\infty$ on T^*. Functions in $\Gamma(T,T^*)$ will be called convex and (T,T^*)-regular.

For example, if $T = V$ is a locally convex topological vector space and $T^* = V^*$—the set of all *continuous* linear functionals on V, then $\Gamma(V,V^*)$ consists of all convex and lower semi-continuous (l.s.c.) functionals on V. More generally, if T^* is a subspace of V^* and T^* separates points of T then $\Gamma(T,T^*)$ consists of all convex and l.s.c. functionals on T. In the most important case of a finite dimensional vector space V, the set $\Gamma(T,T^*)$ is the set of all convex and l.s.c. functions on T (these functions may take the value $+\infty$) whose subgradients belong to T^*. If T^* is convex, then it is enough that gradients, where they exist, belong to T^*.

Definition 4 *We say that a subset U of T is G-invariant if $GU \cap T = U$. We say that a function f on T is G-invariant if it is constant on the sets O_x^T for each $x \in T$.*

Definition 5 *For a subset U of T define $L(U)$ to be the smallest convex, G-invariant (in the sense of Definition 4) subset of T that contains U. For $x \in T$ define $L(x) = L(\{x\})$.*

Now we define the key function, whose convexity properties determine the validity of the generalized Davis convexity theorem. Let

$$\psi_y(x) = \sup_{z\in L(x)} y(z), \quad x \in T,\ y \in V'. \tag{1}$$

The function $\psi_y(x)$ is G-invariant on T in the sense of Definition 4. Therefore, there is a unique G-invariant extension $\widehat{\psi}_y(x)$ from T to C.

THEOREM 1 *Let G be a group acting (possibly nonlinearly) on a convex subset C of a vector space V. Let T be a transversal in the sense of Definition 1 and T^* be an arbitrary subset of V'. Assume that $\widehat{\psi}_y(x)$ is convex on C for any $y \in T^*$. Then a G-invariant function $F : C \to \mathbb{R}\cup\{+\infty\}$ is convex on C if its restriction to T is convex and (T,T^*)-regular.*

PROOF: Let $F : C \to \mathbb{R} \cup \{+\infty\}$ be G-invariant. Let f denote its restriction to T. By assumption $f \in \Gamma(T, T^*)$. So, by Definition 3

$$f(x) = \sup_{y \in T^*} \{y(x) - \alpha(y)\}$$

for some function $\alpha : T^* \to \mathbb{R} \cup \{+\infty\}$. Fix $x_0 \in T$ and consider the set

$$\mathcal{F}_{x_0} = \{x \in T : \ f(x) \leq f(x_0)\}.$$

The set $\mathcal{F}_{x_0}$ is convex, G-invariant in the sense of Definition 4, and contains x_0. Therefore, $L(x_0) \subset \mathcal{F}_{x_0}$. It follows that

$$f(x_0) \leq \sup_{x \in L(x_0)} f(x) \leq \sup_{x \in \mathcal{F}_{x_0}} f(x) = f(x_0).$$

Therefore we have equality everywhere in the chain of inequalities above. In particular,

$$f(x_0) = \sup_{x \in L(x_0)} f(x) = \sup_{x \in L(x_0)} \sup_{y \in T^*} \{y(x) - \alpha(y)\} = \sup_{y \in T^*} \{\psi_y(x_0) - \alpha(y)\}.$$

Since, $x_0 \in T$ was arbitrary, we obtain that for all $x \in C$

$$F(x) = \sup_{y \in T^*} \{\widehat{\psi}_y(x) - \alpha(y)\}.$$

By our assumption, $\widehat{\psi}_y(x)$ is convex on C for all $y \in T^*$. We conclude that $F(x)$ is convex as a supremum of convex functions. ∎

In order to use the theorem one has to pick T first, observing that T has to contain the convex hull of all fixed points of the G-action. Then one has to compute the function $\psi_y(x)$ for all $y \in V'$. To this end one might need to compute $L(x)$ first, which in itself can be a non-trivial task.

Next, one has to determine the subset T^* of V' for which the function $\widehat{\psi}_y(x)$ is convex on C. Verification of convexity might also present technical problems in practice. Finally, once T^* is known, one needs to prove that the restriction to T of one's favorite G-invariant function is in $\Gamma(T, T^*)$.

Our next theorem allows one to avoid the labor-intensive convexity check in special circumstances when the group action is linear.

THEOREM **2** *Assume that the group action is linear. In addition assume that*

$$\sup_{z \in \mathcal{O}_x^T} y(z) = \sup_{z \in \mathcal{O}_x} y(z), \qquad \forall y \in T^* \tag{2}$$

for all $x \in C$. Then

$$\widehat{\psi}_y(x) = \sup_{g \in G} y(gx).$$

Therefore, $\widehat{\psi}_y(x)$ is convex as a supremum of linear functions.

Condition (2) is related to Lewis's so called "normal decomposition system" [12] but is more general in the sense that condition (2) is always satisfied whenever there is a normal decomposition system. (See [12, 13] for a broad discussion of convexity and group invariance in the case of finite dimensional representations of compact Lie groups.)

PROOF: We begin by observing that for any $x \in T$

$$\mathrm{conv}(\mathcal{O}_x^T) \subset L(x) \subset \mathrm{conv}(\mathcal{O}_x) \cap T.$$

Since $L(x)$ is G-invariant in the sense of Definition 4, it follows that $\mathcal{O}_x^T \subset L(x)$. Since $L(x)$ is also convex, we get that $\mathrm{conv}(\mathcal{O}_x^T) \subset L(x)$. The remaining inclusion follows from the fact that $\mathrm{conv}(\mathcal{O}_x) \cap T$ is a convex, G-invariant (for linear group actions only!) subset of T containing x. We therefore, have the corresponding inequalities:

$$\sup_{z \in \mathrm{conv}(\mathcal{O}_x^T)} y(z) \leq \psi_y(x) \leq \sup_{z \in \mathrm{conv}(\mathcal{O}_x) \cap T} y(z) \leq \sup_{z \in \mathrm{conv}(\mathcal{O}_x)} y(z).$$

Next, observe that suprema of linear functions over sets and their convex hulls are the same. Therefore, we get the inequality

$$\sup_{z \in \mathcal{O}_x^T} y(z) \leq \psi_y(x) \leq \sup_{z \in \mathcal{O}_x} y(z).$$

Our assumption (2) turns the last inequalities into equalities, proving that for any $x \in C$ we have $\widehat{\psi}_y(x) = \sup_{g \in G} y(gx)$. ∎

To illustrate Theorem 2, take $V = C = \mathbb{R}^d$, $G = SO(\mathbb{R}^d)$. Let T be a coordinate axis, and let T^* be the set of linear functionals on $\mathbb{R}^d$ obtained by taking inner products with vectors in T. Then the G-orbits are spheres and the restriction of a linear functional $y \in T^*$ to $\mathcal{O}_x$ is maximum or minimum when the orbit intersects T. Thus condition (2) is satisfied and we conclude that a radial lower-semicontinuous function F on $\mathbb{R}^d$ is convex if and only if its restriction to any axis is convex.

We remark that sometimes one can even avoid having to check condition (2) in the context of adjoint action on reductive Lie algebras by applying our Theorems 7–9. We discuss this in more detail in Section 4. Our condition (2) is also a simple consequence of Kostant's convexity theorem [11]. If V is a Lie algebra of a semi-simple Lie group acting on V by the adjoint action and the transversal T is a Cartan subalgebra, then $\mathcal{O}_x^T$ is a discrete set of points permuted by the action of the Weyl group. The set T^* here is identified with T under any G-invariant inner product on V. Kostant's theorem [11] says that the orthogonal projection of $\mathcal{O}_x$ onto T coincides with the convex hull of $\mathcal{O}_x^T$. Under these circumstances, if $y \in T$ then $(y, x) = (y, \pi x)$, where (y, x) denotes the G-invariant inner product of y and x and π denotes the orthogonal projection of x onto T. By Kostant's theorem $\pi x \in \mathrm{conv}(\mathcal{O}_x^T)$, so there exist non-negative numbers λ_ξ that add up to 1 such that

$$\pi x = \sum_{\xi \in \mathcal{O}_x^T} \lambda_\xi \xi.$$

Thus,

$$(y, x) = \sum_{\xi \in \mathcal{O}_x^T} \lambda_\xi (y, \xi).$$

Therefore, there exists $\xi^* \in \mathcal{O}_x^T$ such that $(y, x) \leq (y, \xi^*) \leq \max_{\xi \in \mathcal{O}_x^T} (y, \xi)$. We conclude that condition (2) holds.

Recall from the introduction that we are interested in convex, rotationally invariant sets. However, the Davis convexity theorem talks about convex functions. Unfortunately, there is no exact equivalence between convex sets on a vector space V and convex functions on V. Facts about convex functions on V give more than enough information about convex sets on V, while facts about convex sets on $V \times \mathbb{R}$ give more than enough information about convex functions on V. A convex function f on V can be completely understood in terms of its epigraph $\mathrm{epi}(f) = \{(x, \alpha) \in V \times \mathbb{R} : \ \alpha \geq f(x)\}$, while the level sets $\{x \in V : \ f(x) \leq \alpha\}$ do not contain enough information to characterize a convex function f. At the same time, a convex function gives an adequate description of its level sets. Here is an analogue of the Davis convexity theorem for sets.

THEOREM **3** *Let $L(U)$ be as in Definition 5. Assume that for any two points $\{x_1, x_2\} \subset T$ the set $GL(\{x_1, x_2\})$ is convex. Then for any subset U of C the set $\mathbb{L}(U)$—the smallest G-invariant convex set containing U, is $GL(GU \cap T)$.*

PROOF: Obviously, $GU \cap T \subset \mathbb{L}(U)$, and so $L(GU \cap T) \subset \mathbb{L}(U)$. Therefore, $GL(GU \cap T) \subset \mathbb{L}(U)$. Let us show that the set $GL(GU \cap T)$ is convex. Take any two points $\{x_1, x_2\} \subset GL(GU \cap T)$. Let $x_1^* \in O_{x_1}^T$ and $x_2^* \in O_{x_2}^T$. Obviously, $\{x_1^*, x_2^*\} \subset L(GU \cap T)$. Therefore, $L(\{x_1^*, x_2^*\}) \subset L(GU \cap T)$ and $GL(\{x_1^*, x_2^*\}) \subset GL(GU \cap T)$. But, by assumption, $GL(\{x_1^*, x_2^*\})$ is convex and contains x_1 and x_2, since it contains the G-orbits of x_1^* and x_2^*. Consequently, any convex combination of x_1 and x_2 belongs to $GL(\{x_1^*, x_2^*\})$ and therefore to $GL(GU \cap T)$. Thus, the set $GL(GU \cap T)$ is convex, G-invariant and contains U. It follows that $\mathbb{L}(U) \subset GL(GU \cap T)$, and so $\mathbb{L}(U) = GL(GU \cap T)$. ∎

We can use Theorem 3 to obtain a different generalization of the Davis convexity theorem.

THEOREM **4** *Define an action of G on $C \times \mathbb{R}$ by $g \cdot (x, \alpha) = (gx, \alpha)$. Define $\widehat{T} = T \times \mathbb{R}$. Then $\widehat{T}$ is a transversal in the sense of Definition 1 for the group action on $C \times \mathbb{R}$. For a subset $\widehat{U}$ of $\widehat{T}$ define $\widehat{L}(\widehat{U})$ to be the smallest convex, G-invariant (in the sense of Definition 4) subset of $\widehat{T}$ that contains $\widehat{U}$. Suppose that for every $\{\widehat{x}_1, \widehat{x}_2\} \subset \widehat{T}$ the set $G \cdot \widehat{L}(\{\widehat{x}_1, \widehat{x}_2\})$ is convex. Then a G-invariant function is convex if and only if its restriction to T is convex.*

PROOF: The proof is almost a trivial consequence of Theorem 3 applied for the group action on $C \times \mathbb{R}$ defined in the theorem. It suffices to observe that if f denotes the restriction of a G-invariant function F from C to T, then $\mathrm{epi}(F) = G \cdot \mathrm{epi}(f)$, where

$$\mathrm{epi}(f) = \{(t, \alpha) \in \widehat{T} : \alpha \geq f(t)\}$$

is the epigraph of f. ∎

Theorems 3 and 4 suggest that the basic objects one should study are the sets $GL(\{x_1, x_2\})$. In practice, however, these sets may be very hard to compute. In fact, in most cases in the theory of composite materials the sets $L(\{x_1, x_2\})$ are not known. In particular, they are not known for the 3D conductivity problem described above. For 2D conductivity, the sets $L(\{x_1, x_2\})$ are known and the G-closure problem has been completely solved by Francfort and Milton [5]. In fact, the explicit description of the set $\mathbb{L}(\{x_1, x_2\})$ in [5] shows that $\mathbb{L}(\{x_1, x_2\}) = GL(\{x_1^*, x_2^*\})$ and Theorem 3 applies. But even for 2D conductivity the sets $\widehat{L}(\{\widehat{x}_1, \widehat{x}_2\})$ are not known in general. These sets correspond to polycrystalline G-closures of two crystals with prescribed volume fractions. Quite recently the sets $\widehat{L}(\{\widehat{x}_1, \widehat{x}_2\})$ have been computed for 2D conductivity under the assumption that $\widehat{x}_2$ is a fixed point of the group action [16]. For this reason at present we are unable to give an example of the application of Theorem 4. Our final remark is that there seems to be no apparent relation of Theorem 1 and Theorem 4 in the sense that we cannot show that one is stronger than the other. Conditions of Theorem 4 do not imply that the function $\psi_y(x)$ is convex on T, nor conditions of Theorem 1 imply that the sets $GL(\{x_1, x_2\})$ are convex. Theorem 4 is capable of proving that *every* convex and G-invariant function on T is convex when extended to C. In situations where some convex and G-invariant function on T extend to convex functions and some others—to non-convex functions, Theorem 4 lacks the discriminating power, while Theorem 1 is more flexible. Even so, Theorem 1 can be too crude to establish convexity of individual functions because it only tells us whether or not *all* functions in $\Gamma(T, T^*)$ extend to convex functions on C. For example, we will see in Section 3.2 that in the physically relevant context of 2D conductivity there are convex $SO(2)$-invariant functions whose convexity does not follow from Theorem 1. At the same time the results of [5] show that there is a one-to-one correspondence between convex $SO(2)$-invariant susbsets of T and convex $SO(2)$-invariant susbsets of C.

3 Composite materials

The motivating problem is an effort to understand how the properties of a composite material depend on the properties of its constituents. It is a well-known fact that the properties of a composite depend not only on which materials are used in its construction but also on their geometric arrangement, or the *microstructure*. In many problems of practical interest the microstructure is either unknown or is not controllable. Therefore, a natural thing to ask is to describe the set of properties of composites, called *effective properties*, made with the given materials as the microstructure is varying over all possible configurations. This problem is called the G-closure problem, first introduced in [14, 19].

3.1 3D conducting polycrystals

In the context of heat conduction or electrostatics, the relevant properties of a material are described by a 3×3 symmetric and positive definite matrix, which is called the *conductivity tensor*. The set of 3×3 symmetric and positive definite matrices will be denoted $\mathrm{Sym}(\mathbb{R}^3)^+$.

A very common type of composite material is a *polycrystal*. A polycrystal is made of many grains of a single crystal with the conductivity tensor $\boldsymbol{L}_0$. The grains are characterized by the particular orientation of the crystal in space. Therefore, we say that the set of constituent materials for the polycrystal is the set

$$\mathcal{U} = \{\boldsymbol{R}\boldsymbol{L}_0\boldsymbol{R}^t : \ \boldsymbol{R} \in SO(3)\}.$$

Another common example is a *multiphase composite* with isotropic constituents. The set of constituents $\mathcal{U}$ in this case has the form

$$\mathcal{U} = \{\alpha_1\boldsymbol{I}, \alpha_2\boldsymbol{I}, \ldots, \alpha_r\boldsymbol{I}\},$$

where $\boldsymbol{I}$ is the 3×3 identity matrix. The corresponding set of the conductivity tensors of effective properties of composites made with materials in $\mathcal{U}$ is called the *G-closure* of $\mathcal{U}$ and is denoted by $G(\mathcal{U})$. The G-closure problem, therefore, consists in determining $G(\mathcal{U})$ from the set $\mathcal{U}$. An elementary property of G-closures is that the set $G(\mathcal{U})$ is $SO(3)$-invariant, whenever the set $\mathcal{U}$ is. In the two examples above the set $\mathcal{U}$ is $SO(3)$-invariant.

In a 1980 paper [15], Milton derived an important geometric property of the set $G(\mathcal{U})$, namely that for any unit vector $\boldsymbol{n} \in \mathbb{R}^3$, the set $W_{\boldsymbol{n}}(G(\mathcal{U}))$ is convex, where

$$W_{\boldsymbol{n}}(\boldsymbol{L}) = [(\boldsymbol{I} - \boldsymbol{L})^{-1} - \boldsymbol{n} \otimes \boldsymbol{n}]^{-1}. \tag{3}$$

Physically, this condition can be formulated as follows. If we take $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ from $G(\mathcal{U})$ and make a composite by layering these materials in alternating layers with normal $\boldsymbol{n}$ then the effective tensor of such a composite must belong to $G(\mathcal{U})$. Varying the volume fraction of material $\boldsymbol{L}_1$ from 0 to 1 will produce the effective tensors tracing the straight segment from $W_{\boldsymbol{n}}(\boldsymbol{L}_1)$ to $W_{\boldsymbol{n}}(\boldsymbol{L}_2)$ in the $W_{\boldsymbol{n}}(\boldsymbol{L})$ space. Therefore, we can consider a geometric problem of computing the set $\mathcal{L}(\mathcal{U})$, which is the smallest set containing the set $\mathcal{U}$ that satisfies the convexity property of Milton. Physically, the set $\mathcal{L}(\mathcal{U})$ is the closure of the set of all effective tensors of multi-rank laminates (laminates of laminates, etc.) made with materials in $\mathcal{U}$. In the case of an $SO(3)$-invariant set $\mathcal{U}$, Milton's convexity condition can be simplified by fixing $\boldsymbol{n} = \boldsymbol{e}_1 = (1, 0, 0)$ and asking to find the smallest $SO(3)$-invariant set $\mathcal{L}(\mathcal{U})$ containing the set $\mathcal{U}$ such that $W(\mathcal{L}(\mathcal{U}))$ is convex, where $W(\boldsymbol{L}) = W_{\boldsymbol{e}_1}(\boldsymbol{L})$. We prove this assertion in Theorem 5 below.

Our idea is that it may be more advantageous to work in the $\boldsymbol{K} = W(\boldsymbol{L})$ space of variables than in the physical space of $\boldsymbol{L}$ variables. For example, it was the key idea for the recently developed theory of exact relations for composites [6, 7, 8, 9]. In the $\boldsymbol{K}$-space we have a problem of finding, for a given subset U of

$$C = W(\mathrm{Sym}^+(\mathbb{R}^3)) = \left\{\boldsymbol{A} \in \mathrm{Sym}(\mathbb{R}^3) : k_{11} > -1, \ k_{22} < 1, \ (1 - k_{22})(1 - k_{33}) > k_{23}^2\right\},$$

a smallest convex subset $\mathbb{L}(U)$ of C containing U and such that the sets $W_{\boldsymbol{n}}(W^{-1}(\mathbb{L}(U)))$ are convex for all unit vectors $\boldsymbol{n} \in \mathbb{R}^3$. Observe that $W_{\boldsymbol{n}}(W^{-1}(\boldsymbol{K})) = \Lambda_{\boldsymbol{A}(\boldsymbol{n})}(\boldsymbol{K})$, where

$$\Lambda_{\boldsymbol{A}}(\boldsymbol{K}) = (\boldsymbol{I} - \boldsymbol{K}\boldsymbol{A})^{-1}\boldsymbol{K}$$

and

$$\boldsymbol{A}(\boldsymbol{n})=\boldsymbol{n}\otimes\boldsymbol{n}-\boldsymbol{e}_1\otimes\boldsymbol{e}_1.$$

The map $\Lambda_{\boldsymbol{A}}$ has several interesting properties whose relevance to the question above is not clear.

- $\Lambda_{\boldsymbol{A}}(\Lambda_{\boldsymbol{B}}(\boldsymbol{K}))=\Lambda_{\boldsymbol{A}+\boldsymbol{B}}(\boldsymbol{K})$.
- $\Lambda_{\boldsymbol{A}}(\boldsymbol{K}+\boldsymbol{K}_0)=\Lambda_{\boldsymbol{A}}(\boldsymbol{K}_0)+\Lambda_{\boldsymbol{A}'}(\boldsymbol{Q}\boldsymbol{K}\boldsymbol{Q}^t)$, where $\boldsymbol{A}'=\boldsymbol{A}-\boldsymbol{A}\boldsymbol{K}_0\boldsymbol{A}$ and $\boldsymbol{Q}=(\boldsymbol{I}-\boldsymbol{K}_0\boldsymbol{A})^{-1}$.
- $\Lambda_{\boldsymbol{A}}(\boldsymbol{K})=\boldsymbol{K}+\boldsymbol{K}\boldsymbol{A}\boldsymbol{K}+\ldots+(\boldsymbol{K}\boldsymbol{A})^n\boldsymbol{K}+\ldots$

Now let us prove that in the case of an $SO(3)$-invariant subset $\mathcal{U}\subset\mathrm{Sym}^+(\mathbb{R}^3)$ we need to solve a "simplified" problem mentioned above.

THEOREM 5 *Suppose a subset S of $\mathrm{Sym}^+(\mathbb{R}^3)$ is $SO(3)$-invariant. Let $\mathcal{L}(S)$ denote the smallest $SO(3)$-invariant subset of $\mathrm{Sym}^+(\mathbb{R}^3)$ containing S such that $W(\mathcal{L}(S))$ is convex. Let $U=W(S)$ and let $\mathbb{L}(U)$ denote the smallest convex subset of $C=W(\mathrm{Sym}^+(\mathbb{R}^3))$ containing U and such that $\Lambda_{\boldsymbol{A}(\boldsymbol{n})}(\mathbb{L}(U))$ is convex for all $\boldsymbol{n}\in\mathbb{S}^2$. Then*

$$W(\mathcal{L}(S))=\mathbb{L}(U). \tag{4}$$

PROOF: Observe that for any rotation $\boldsymbol{R}\in SO(3)$ we have

$$\boldsymbol{R}W_{\boldsymbol{n}}(\boldsymbol{L})\boldsymbol{R}^t=W_{\boldsymbol{R}\boldsymbol{n}}(\boldsymbol{R}\boldsymbol{L}\boldsymbol{R}^t). \tag{5}$$

We also have that

$$\Lambda_{\boldsymbol{A}(\boldsymbol{n})}=W_{\boldsymbol{n}}\circ W^{-1}. \tag{6}$$

Then we can say that since the set $\mathcal{L}(S)$ is rotationally invariant, it follows that for any $\boldsymbol{n}\in\mathbb{S}^2$ the set $W_{\boldsymbol{n}}(\mathcal{L}(S))$ is convex, but $W_{\boldsymbol{n}}(\mathcal{L}(S))=\Lambda_{\boldsymbol{A}(\boldsymbol{n})}(W(\mathcal{L}(S)))$. Thus, the set $W(\mathcal{L}(S))$ contains U and has the property that $\Lambda_{\boldsymbol{A}(\boldsymbol{n})}(W(\mathcal{L}(S)))$ is convex for all $\boldsymbol{n}\in\mathbb{S}^2$. Hence, $\mathbb{L}(U)\subset W(\mathcal{L}(S))$.

To get the reverse inclusion we need to show that the set $W^{-1}(\mathbb{L}(W(S))$ is rotationally invariant. In order to prove this we need to realize that $\mathbb{L}(U)$ can be constructed as follows:

$$\mathbb{L}(U)=\bigcup_{k=0}^{\infty}U_k\equiv\widehat{U}, \tag{7}$$

where $U_0=U$ and

$$U_{k+1}=\bigcup_{\boldsymbol{n}\in\mathbb{S}^2}\Lambda^{-1}_{\boldsymbol{A}(\boldsymbol{n})}(\mathrm{conv}(\Lambda_{\boldsymbol{A}(\boldsymbol{n})}(U_k))). \tag{8}$$

Indeed, it is clear that $\widehat{U}\subset\mathbb{L}(U)$. But also for any $\boldsymbol{n}\in\mathbb{S}^2$ and any $\{y_1,y_2\}\subset\Lambda_{\boldsymbol{A}(\boldsymbol{n})}(\widehat{U})$ there exists $k\ge 0$ such that $\{y_1,y_2\}\subset\Lambda_{\boldsymbol{A}(\boldsymbol{n})}(U_k)$. Thus, $y_\lambda=\lambda y_1+(1-\lambda)y_2\in\mathrm{conv}(\Lambda_{\boldsymbol{A}(\boldsymbol{n})}(U_k))$. Therefore, $\Lambda^{-1}_{\boldsymbol{A}(\boldsymbol{n})}(y_\lambda)\in U_{k+1}\subset\widehat{U}$. So $y_\lambda\in\Lambda_{\boldsymbol{A}(\boldsymbol{n})}(\widehat{U})$ and $\Lambda_{\boldsymbol{A}(\boldsymbol{n})}(\widehat{U})$ is convex for all $\boldsymbol{n}\in\mathbb{S}^2$. Also, $U\subset\widehat{U}$. Therefore, $\mathbb{L}(U)\subset\widehat{U}$, and (7) is proved.

Now, in order to show that $\widehat{S}=W^{-1}(\mathbb{L}(U))$ is $SO(3)$-invariant we denote $S_k=W^{-1}(U_k)$ and observe that

$$\widehat{S}=\bigcup_{k=0}^{\infty} S_k, \quad S_0=S,$$

$$S_{k+1}=\bigcup_{\boldsymbol{n}\in\mathbb{S}^2} W_{\boldsymbol{n}}^{-1}(\mathrm{conv}(W_{\boldsymbol{n}}(S_k))).$$

The last relation is just (8) with $\Lambda_{\boldsymbol{A}(\boldsymbol{n})}$ replaced by its definition (6). We claim that each set S_k is $SO(3)$-invariant. We prove this by induction. $S_0=S$ is $SO(3)$-invariant by assumption. Now assume that S_k is $SO(3)$-invariant. If $\boldsymbol{L}\in S_{k+1}$ then there exist $\boldsymbol{n}\in\mathbb{S}^2$, $\{\boldsymbol{L}_1,\ldots,\boldsymbol{L}_p\}\subset S_k$ and non-negative numbers $\lambda_1,\ldots,\lambda_p$ that add up to 1 such that

$$\boldsymbol{L}=W_{\boldsymbol{n}}^{-1}\left(\sum_{j=1}^{p}\lambda_j W_{\boldsymbol{n}}(\boldsymbol{L}_j)\right).$$

It follows now from (5) that

$$\boldsymbol{R}\boldsymbol{L}\boldsymbol{R}^t=W_{\boldsymbol{R}\boldsymbol{n}}^{-1}\left(\sum_{j=1}^{p}\lambda_j W_{\boldsymbol{R}\boldsymbol{n}}(\boldsymbol{R}\boldsymbol{L}_j\boldsymbol{R}^t)\right).$$

Therefore, $\boldsymbol{R}\boldsymbol{L}\boldsymbol{R}^t\in S_{k+1}$, since by the inductive hypothesis $\boldsymbol{R}\boldsymbol{L}_j\boldsymbol{R}^t\in S_k$. This finishes the proof that S_k, and as a consequence, $\widehat{S}$ is $SO(3)$-invariant. Thus, (4) is proved. ∎

In general, the computation of the G-closure sets is very complicated, but in the few cases where the answers have been obtained it was true that $\mathcal{L}(\mathcal{U})=G(\mathcal{U})$. In his book [17, Section 31.9] Milton gives an example showing that the sets $\mathcal{L}(\mathcal{U})$ and $G(\mathcal{U})$ are in general, different. Nevertheless, we believe that the set $\mathcal{L}(\mathcal{U})$ gives a good idea of what the set $G(\mathcal{U})$ should look like. For two-dimensional conductivity the set $G(\mathcal{U})=\mathcal{L}(\mathcal{U})$ can be described for general sets $\mathcal{U}$ by a construction algorithm, terminating after two steps, that starts with the set $\mathcal{U}$ and on each step transforms the current set by computing the convex hull in a different coordinate system. The final set in the algorithm is $G(\mathcal{U})$. This algorithm has been found by Francfort and Milton [5]. In particular, the whole construction in [5] occurs on the set of diagonal 2×2 matrices. This brings us to the idea that in 3D the convexity of $\mathcal{L}(\mathcal{U})$ can somehow be inferred by looking at the set of diagonal matrices or another subspace transversal to $SO(3)$ orbits. If we are to use Theorem 3 then we need to solve the following two problems. These problems are still open. They are of fundamental importance in the theory of composites. At the same time they may appeal to broader mathematical audience since they are really geometric, in essence.

Problem 1. Let $(\mathbb{R}^3)^+=\{(a_1,a_2,a_3): a_j>0,\ j=1,2,3\}$. We define a *non-linear* action of the permutation group S_3 on $(\mathbb{R}^3)^+$. If $\sigma\in S_3$ and $\boldsymbol{a}\in(\mathbb{R}^3)^+$, then $\sigma(\boldsymbol{a})$ denotes the standard linear action of S_3 on $(\mathbb{R}^3)^+$. Let

$$j(\boldsymbol{x})=\left(\frac{1}{x_1},x_2,x_3\right).$$

Then we define the non-linear action of S_3 on $(\mathbb{R}^3)^+$ by

$$\sigma \cdot \boldsymbol{x} = (j \circ \sigma \circ j)(\boldsymbol{x}).$$

The first problem is the following. For $\{\boldsymbol{a}, \boldsymbol{b}\} \subset (\mathbb{R}^3)^+$ find $L(\boldsymbol{a}, \boldsymbol{b})$—the smallest subset of $(\mathbb{R}^3)^+$ such that

1. $\{\boldsymbol{a}, \boldsymbol{b}\} \subset L(\boldsymbol{a}, \boldsymbol{b})$,
2. $L(\boldsymbol{a}, \boldsymbol{b})$ is convex,
3. $L(\boldsymbol{a}, \boldsymbol{b})$ is S_3-invariant.

Problem 2 (Conjecture). Now define the map $J : (\mathbb{R}^3)^+ \to \mathrm{Sym}(\mathbb{R}^3)$ by

$$J(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{x_1} & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}.$$

We conjecture that $\forall \{\boldsymbol{a}, \boldsymbol{b}\} \subset (\mathbb{R}^3)^+$ the set $W(SO(3)J(L(\boldsymbol{a}, \boldsymbol{b})))$ is convex. Here the map W is defined by (3) with $\boldsymbol{n} = \boldsymbol{e}_1$.

Our conjecture is equivalent (via Theorem 3) to saying that $W(\mathcal{U})$ is convex for an $SO(3)$-invariant set $\mathcal{U} \subset \mathrm{Sym}^+(\mathbb{R}^3)$ if and only if $W(\mathcal{U} \cap T)$ is convex, where T is the set of diagonal matrices.

We remark that the set $L(\boldsymbol{a}, \boldsymbol{b})$ is not known even for the simplest case when $\boldsymbol{b} = \boldsymbol{a} = (a_1, a_2, a_2)$. Avellaneda, Cherkaev, Lurie and Milton [2] have computed part of the boundary of $G(J(\{\boldsymbol{a}\}))$ (see [2, FIG. 6]). They also conjectured that the remaining part may have a non-algebraic form (see [2, Equation (81)]). Their conjecture can be reformulated as a conjecture on the shape of $L(\boldsymbol{a}, \boldsymbol{a}) \subset (\mathbb{R}^3)^+$ under the assumption that $1/a_1 < a_2 = a_3$. In our notation the intersection of $L(\boldsymbol{a}, \boldsymbol{a})$ with the part of the surface $x_1 x_2 = 1$, where $x_3 > x_2$ is conjectured to be the curve

$$x_3 = a_2 - (a_2 - a_0) \left[\frac{(x_2 - \sqrt{a_2/a_1})(a_0 + \sqrt{a_2/a_1})}{(x_2 + \sqrt{a_2/a_1})(a_0 - \sqrt{a_2/a_1})} \right]^{1/\sqrt{a_1 a_2}}, \quad x_1 = \frac{1}{x_2},$$

where

$$a_0 = \frac{\sqrt{1 + 8a_1 a_2} - 1}{2a_1}.$$

Since problems 1 and 2 are open at the moment, we conclude this section with an application of Theorem 1. Before attempting to apply the theorem to 3D conductivity we decided to check what it gives us in the case of 2D conductivity, where the answers are already known [5]. We apply Theorem 1 to the case of 2D conductivity, pretending that the results of [5] are not available to us. Our conclusion is that Theorem 1 is not sufficient to describe convex, $SO(2)$-invariant hulls of sets because in our situation Theorem 1 is not delicate enough to characterize all convex, $SO(2)$-invariant functions. Nevertheless, we believe that the example that we work out below is enlightening.

3.2 2D conducting polycrystals

The space V in the case of 2D conductivity is the space of all symmetric 2×2 matrices. Let P be the set of all 2×2 symmetric positive definite matrices. Let W be a non-linear map from P to V given by (3) (with $\boldsymbol{n}=\boldsymbol{e}_1$). The map W is smooth and injective on P. It is known from physical considerations that the set P is G-closed. Therefore, the set $C=W(P)$ is a convex subset of V. In fact the set C can be described explicitly:

$$C=\left\{\boldsymbol{K}=\begin{bmatrix}k_{11} & k_{12}\\ k_{12} & k_{22}\end{bmatrix}\ :\ k_{11}>-1,\ k_{22}<1\right\}.$$

The group $SO(2)$ acts on C according to the formula

$$\boldsymbol{R}\cdot\boldsymbol{K}=W(\boldsymbol{R}W^{-1}(\boldsymbol{K})\boldsymbol{R}^t),\ \ \boldsymbol{K}\in C. \tag{9}$$

We observe that the two-dimensional transversal T has to lie in the set of diagonal matrices because it has to contain the convex hull of fixed points $\boldsymbol{K}_\alpha=W(\alpha\boldsymbol{I})$ of the group action. Therefore, T must contain the convex hull of the hyperbola $\{\boldsymbol{K}\in C:\ (1+k_{11})(1-k_{22})=1,\ k_{12}=0\}$. Thus, one possible choice for T is

$$T=\{\boldsymbol{K}\in C:\ (1+k_{11})(1-k_{22})\geq 1,\ k_{12}=0\}. \tag{10}$$

Another possible choice for T is the full quadrant

$$T=\{\boldsymbol{K}\in C:\ k_{11}>-1,\ k_{22}<1,\ k_{12}=0\}. \tag{11}$$

THEOREM **6** *Suppose T is given either by (10) or by (11). Then the set of matrices $\boldsymbol{Y}$ for which $\widehat{\psi}_{\boldsymbol{Y}}(\boldsymbol{K})$ is convex is $T^*=\{\boldsymbol{Y}:y_{11}\geq 0,\ y_{22}\leq 0\}$.*

PROOF: According to Theorem 1, we need to compute

$$\widehat{\psi}_{\boldsymbol{Y}}(\boldsymbol{K})=\max_{\boldsymbol{K}^*\in L(O^T_{\boldsymbol{K}})}(\boldsymbol{K}^*,\boldsymbol{Y}), \tag{12}$$

where we use a convenient inner product $(\boldsymbol{A},\boldsymbol{B})=\mathrm{Tr}\,(\boldsymbol{AB})$ on $V=\mathrm{Sym}(\mathbb{R}^2)$. If we make a choice of T according to (10) then for each $\boldsymbol{K}\in C$ the set $O^T_{\boldsymbol{K}}$ contains a single point $\boldsymbol{K}^*(\boldsymbol{K})$ and is therefore convex. So, $\widehat{\psi}_{\boldsymbol{Y}}(\boldsymbol{K})=(\boldsymbol{K}^*(\boldsymbol{K}),\boldsymbol{Y})$.

LEMMA **1** *Suppose T is given by (10). Then*

$$\widehat{\psi}_{\boldsymbol{Y}}(\boldsymbol{K})=y_{22}-y_{11}+\psi_0\left(\Delta+\sqrt{\Delta^2-4\lambda}\right)\Big/2\lambda, \tag{13}$$

where

$$\begin{aligned}\psi_0\ &=y_{11}a_{11}-y_{22}a_{22},\\ \Delta\ &=\lambda+1+k_{12}^2,\\ a_{11}&=1+k_{11},\\ a_{22}&=1-k_{22},\\ \lambda\ &=a_{11}a_{22}.\end{aligned}$$

PROOF: It is quite easy to find $\boldsymbol{K}^*(\boldsymbol{K})$. The $SO(2)$ orbit of $\boldsymbol{K}$ is described by the equations

$$\frac{a_{22}^*}{a_{11}^*}=\frac{a_{22}}{a_{11}},\qquad \frac{\det \boldsymbol{K}^*}{a_{11}^*}=\frac{\det \boldsymbol{K}}{a_{11}}. \tag{14}$$

We have

$$\det \boldsymbol{K}^*=(a_{11}^*-1)(1-a_{22}^*),$$

since $\boldsymbol{K}^*\in T$ and $k_{12}^*=0$. Replacing $\det \boldsymbol{K}$ by $a_{11}+a_{22}-\Delta$ in (14) and solving the equations for a_{11}^* and a_{22}^* we obtain:

$$a_{11}^*=(\Delta\pm\sqrt{\Delta^2-4\lambda})/2a_{22},\quad a_{22}^*=(\Delta\pm\sqrt{\Delta^2-4\lambda})/2a_{11}.$$

The plus sign corresponds to $\boldsymbol{K}^*\in T$, because then $a_{11}^*a_{22}^*>1$.

Hence,

$$\widehat{\psi}_{\boldsymbol{Y}}(\boldsymbol{K})=(\boldsymbol{K}^*,\boldsymbol{Y})=y_{22}-y_{11}+\psi_0(\Delta+\sqrt{\Delta^2-4\lambda})/2\lambda.$$

∎

If T is given by (11), then a similar analysis shows that

$$\widehat{\psi}_{\boldsymbol{Y}}(\boldsymbol{K})=y_{22}-y_{11}+(\psi_0\Delta+|\psi_0|\sqrt{\Delta^2-4\lambda})/2\lambda.$$

Let us fix $\boldsymbol{Y}$ and observe that for any $\mu>0$ the convexity of $\widehat{\psi}_{\boldsymbol{Y}}(a_{11},a_{22},k_{12})$ is the same as the convexity of $\widehat{\psi}_{\boldsymbol{Y}}(\mu a_{11},a_{22}/\mu,k_{12})$. Moreover, the hyperbolic rotation $(a_{11},a_{22})\mapsto(\mu a_{11},a_{22}/\mu)$ maps T into T. Therefore, choosing $\mu>0$ appropriately we may reduce ψ_0 to a positive multiple of either $a_{11}+a_{22}$, $a_{11}-a_{22}$ or $a_{22}-a_{11}$.

We will now investigate convexity of $\widehat{\psi}_{\boldsymbol{Y}}(a_{11},a_{22},k_{12})$ by computing its Hessian. For simplicity of notation we denote a_{11} by x, a_{22} by y and k_{12} by z. Then we need to study convexity of

$$\psi(x,y,z)=(\pm x\pm y)(z^2+xy+1+R)/2xy,$$

where

$$R=\sqrt{(z^2+xy+1)^2-4xy}.$$

Let

$$f(x,y,z)=(z^2+xy+1+R)/2y.$$

Then $\psi(x,y,z)=\pm f(x,y,z)\pm f(y,x,z)$. We see that it will be helpful to compute the Hessian of f. The computation is tedious but straightforward. It is convenient to write the answer in terms of the functions R and $g=f/x$ because they do not change if we interchange

x and y.

$$\begin{aligned}
f_{xx} &= \frac{2z^2 y}{R^3}, \\
f_{xy} &= \frac{2z^2 x}{R^3}, \\
f_{xz} &= 2z\,\frac{z^2 - xy + 1}{R^3}, \\
f_{zz} &= 2x\,\frac{gR^2 - 4z^2}{R^3}, \\
f_{yz} &= 2xz\,\frac{z^2 - xy + 1 - gR^2}{yR^3}, \\
f_{yy} &= 2x\,\frac{(z^2+1)R^2 g - R^2 + z^2 xy}{y^2 R^3}.
\end{aligned} \tag{15}$$

Using these expressions it is not difficult to verify that $f_{xx} > 0$ and that $f_{xx}f_{zz} - f_{xz}^2 > 0$, while the determinant of the Hessian vanishes. It remains to verify convexity at the points of the hyperbola $xy = 1$, $z = 0$ where ∇f has a discontinuity. We easily see that

$$f(x,y,z) \geq f(x,y,0) = \begin{cases} x, & xy > 1 \\ 1/y, & xy < 1. \end{cases}$$

For any point $(x_0, 1/x_0, 0)$ on the hyperbola, the hyperplane $P = x$ passes through the point $(x_0, 1/x_0, 0, f(x_0, 1/x_0, 0))$ and lies below the graph of f. Indeed, $f(x,y,z) \geq f(x,y,0) \geq x$, because on $xy < 1$, we have $x < 1/y$. Thus, the hyperplane $P = x$ is a supporting hyperplane at $(x_0, 1/x_0, 0, f(x_0, 1/x_0, 0))$. We conclude that the function $\psi(x,y,z) = f(x,y,z) + f(y,x,z)$ is convex on C. Formulas (15) allow us to show that functions $f(x,y,z) - f(y,x,z)$, $-f(x,y,z) + f(y,x,z)$ and $-f(x,y,z) - f(y,x,z)$ are not convex in any open subregion of C. Thus, the set of matrices $\boldsymbol{Y}$, for which $\widehat{\psi}_{\boldsymbol{Y}}$ is convex on C is $T^* = \{\boldsymbol{Y} : y_{11} \geq 0,\ y_{22} \leq 0\}$.

If we take T to be given by (11), the result is the same: the set of matrices $\boldsymbol{Y}$, for which $\widehat{\psi}_{\boldsymbol{Y}}$ is convex on C is $T^* = \{\boldsymbol{Y} : y_{11} \geq 0,\ y_{22} \leq 0\}$ and for those values of $\boldsymbol{Y}$ the functions $\widehat{\psi}_{\boldsymbol{Y}}$ corresponding to T given by (10) and (11) coincide. ∎

As we remarked after Definition 3, the functions in $\Gamma(T, T^*)$ are all convex, l.s.c. functions on T whose gradients (where they exist) lie in T^*. Using lines of the form $a_{11}y_{11} - a_{22}y_{22}$ =const in (a_{11}, a_{22}) plane with $(y_{11}, y_{22}) \in T^*$ allows us to find some convex and rotationally invariant sets. Since T^* is not the full space $\mathbb{R}^2$ we will not be able to recover the results of Francfort and Milton [5]. Figures 1 and 2 show the smallest sets produced by our theory in (a_{11}, a_{22}) plane that contain two given points A and B.

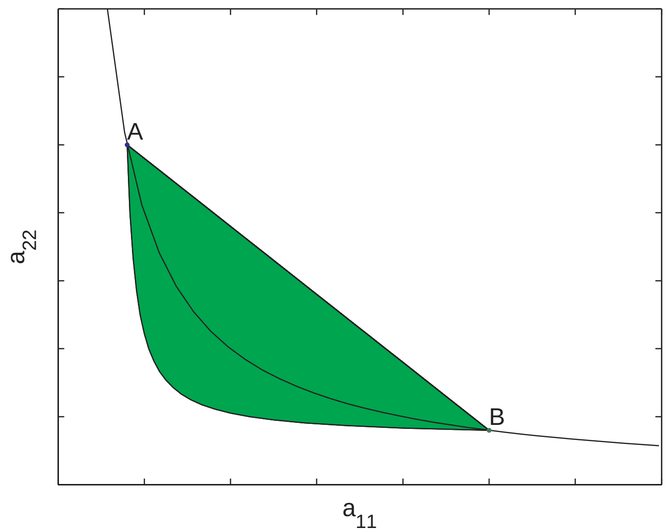

Figure 1: The smallest set produced by our theory that contains A and B, that is convex and rotationally invariant. It coincides with the G-closure of two isotropic conductors.

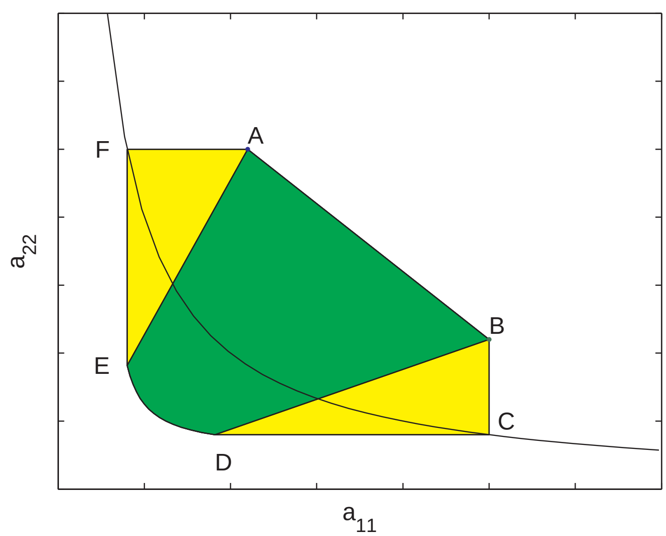

Figure 2: The set $ABCDEF$ is the smallest set produced by our theory that contains A and B, that is convex and rotationally invariant. The set $ABDE$ is the smallest convex, rotationally invariant set containing A and B.

4 Linear Group Actions

In this section, we restrict our attention to linear actions of compact groups.

THEOREM 7 *Let K be a compact Lie group with Lie algebra $\mathfrak{k}$, let $\pi : K \to SO(V)$ be an orthogonal representation on a finite-dimensional euclidean vector space V, and let $T \subset V$ be a subspace. Assume that for a dense set of vectors $y \in T$,*

$$\pi(\mathfrak{k})y = T^{\perp}. \tag{16}$$

Then a K-invariant lower-semicontinuous function F is convex on V if and only if its restriction f is convex on T.

Morally, condition (16) amounts to saying that the tangent space of the orbit of $y \in T$ equals $T^{\perp}$.

PROOF: We establish (2) with T^* equal to all linear functionals on V of the form $x \mapsto \langle y, x\rangle$, $y \in T$. We identify T^* with T. By compactness, it is enough to establish (2) for a dense set of y's in T. Then (2) states that the maximum value of $x \mapsto \langle y, x\rangle$ over $\mathcal{O}_x$ is attained in T for any $y \in T$. In fact x is a critical point of $\langle y, x\rangle \restriction \mathcal{O}_x$ if and only if $\langle y, \pi(X)x\rangle = 0$ for all X in the Lie algebra $\mathfrak{k}$; by (16), this implies $x \in T$. ∎

Below we use Theorem 7 to derive Lewis's [13] Lie-algebraic generalization of Davis's theorem, then we use it to show that the Pfaffian is log-concave. First we recall the Lie-algebra background.

Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra and let $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ be given by the Lie bracket $\mathrm{ad}(X)Y = [X, Y]$. Then $\mathrm{ad}(X)$ is a derivation (Jacobi identity),

$$\mathrm{ad}(X)[Y, Z] = [\mathrm{ad}(X)Y, Z] + [Y, \mathrm{ad}(X)Z], \qquad X, Y, Z \in \mathfrak{g};$$

rewriting this as

$$\mathrm{ad}([X, Y]) = \mathrm{ad}(X)\mathrm{ad}(Y) - \mathrm{ad}(Y)\mathrm{ad}(X) \equiv [\mathrm{ad}(X), \mathrm{ad}(Y)]$$

shows that $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra representation. Let $B(X, Y) = \mathrm{Trace}(\mathrm{ad}(X)\mathrm{ad}(Y))$ be the Killing form; then B is ad-invariant,

$$B(\mathrm{ad}(X)Y, Z) = -B(Y, \mathrm{ad}(X)Z), \qquad X, Y, Z \in \mathfrak{g}.$$

Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$, $\mathfrak{z} = \{X : \mathrm{ad}(X) = 0\}$.

A Lie algebra $\mathfrak{g}$ is *reductive* if it can be decomposed into a vector space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that the Killing form is positive semi-definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$, and

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

This is the Cartan decomposition of $\mathfrak{g}$.

Define a semi-definite euclidean inner product on $\mathfrak{g}$ by setting $\langle X, Y\rangle = B(X, Y)$ on $\mathfrak{p}$, $\langle X, Y\rangle = -B(X, Y)$ on $\mathfrak{k}$, and by insisting that $\mathfrak{k}$ and $\mathfrak{p}$ be orthogonal. Then the operator $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is skew-adjoint for $X \in \mathfrak{k}$ and self-adjoint for $X \in \mathfrak{p}$. Let $\mathfrak{so}(\mathfrak{g})$ denote the skew-adjoint operators on $\mathfrak{g}$, and let $SO(\mathfrak{g})$ denote the group of operators on $\mathfrak{g}$ preserving this inner product and having unit determinant.

If $X \in \mathfrak{p}$ satisfies $B(X, X) = 0$, then $\mathrm{ad}(X)$ is self-adjoint and $\mathrm{Trace}(\mathrm{ad}(X)^2) = 0$. By Cauchy-Schwartz, this implies[1] $\mathrm{ad}(X) = 0$; thus $X \in \mathfrak{z}$. Conversely, since B is negative definite on $\mathfrak{k}$, it follows that $\mathfrak{z} \subset \mathfrak{p}$, hence $\mathfrak{z} = \{X \in \mathfrak{p} : B(X, X) = 0\}$.

A Lie algebra $\mathfrak{k}$ is *compact* if its Killing form is negative definite. Then $\mathfrak{k}$ is compact if and only if there is exactly one connected simply-connected compact Lie group K with Lie algebra $\mathfrak{k}$. Since $\mathrm{ad} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{g})$ is a representation, it lifts uniquely to a group representation $\mathrm{Ad} : K \to SO(\mathfrak{g})$ satisfying $\mathrm{Ad}(\exp(X)) = \exp(\mathrm{ad}(X))$ for $X \in \mathfrak{k}$.

Let K be any compact Lie group with Lie algebra $\mathfrak{k}$ and equipped with a representation $\mathrm{Ad} : K \to SO(\mathfrak{g})$ satisfying $\mathrm{Ad}(\exp(X)) = \exp(\mathrm{ad}(X))$ for $X \in \mathfrak{k}$.

THEOREM 8 *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a reductive Lie algebra, let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let K be any compact Lie group corresponding to $\mathfrak{k}$ as above. Then an $Ad(K)$-invariant function on $\mathfrak{p}$ is convex on $\mathfrak{a}$ if and only if it is convex on $\mathfrak{p}$.*

Choosing $\mathfrak{g} = \mathfrak{gl}(\mathbf{R}^d)$, $\mathfrak{k} = \mathfrak{so}(\mathbf{R}^d)$, $\mathfrak{p} = \mathrm{Sym}(\mathbf{R}^d)$, $\mathrm{Ad}(g)(X) = gXg^{-1}$, $K = SO(\mathbf{R}^d)$, and $\mathfrak{a}$ the diagonal matrices, yields Davis's theorem.

This theorem was derived by Lewis [13] for $\mathfrak{g}$ semisimple using the Kostant convexity theorem [11], which states that the projection P of $\mathcal{O}_x$ onto T is the convex hull of $\mathcal{O}_x^T$. Our condition (2) is weaker than this, as it is equivalent to the equality of the convex hulls of $P(\mathcal{O}_x)$ and $\mathcal{O}_x^T$. Kostant's theorem, as generalized by Atiyah and Guillemin-Sternberg [1, 10], depends on the symplectic structure of the orbits; Theorem 7 above makes no geometric assumption on the orbits other than transversality with T.

PROOF: Since $\mathfrak{a}$ is maximal abelian, $\mathfrak{z} \subset \mathfrak{a}$. Since $\mathrm{ad}(\mathfrak{a})$ is a commuting set of self-adjoint operators on $\mathfrak{g}$, they can be simultaneously diagonalized.[2] By decomposing each eigenvector over $\mathfrak{k} \oplus \mathfrak{p}$, there is a finite set Σ of nonzero linear functionals λ (possibly repeated) in $\mathfrak{a}^*$ and nonzero vectors $X_\lambda \in \mathfrak{k}$, $P_\lambda \in \mathfrak{p}$, such that with $\mathfrak{k}_\lambda = \mathbf{R}X_\lambda$, $\mathfrak{p}_\lambda = \mathbf{R}P_\lambda$, we have

$$[A, X_\lambda] = \lambda(A)P_\lambda \text{ and } [A, P_\lambda] = \lambda(A)X_\lambda, \qquad A \in \mathfrak{a}, \lambda \in \Sigma,$$

and

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{k}_\lambda \text{ and } \mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{p}_\lambda$$

is an orthogonal direct sum. Here $\mathfrak{k}_0$ is the set of elements in $\mathfrak{k}$ commuting with elements of $\mathfrak{a}$.

[1]Cauchy-Schwartz holds for semi-definite inner products.

[2]If $(V, \langle\cdot,\cdot\rangle)$ is a semi-definite real inner product space and $A : V \to V$ is self-adjoint and satisfies $A(V^\perp) = 0$, where $V^\perp = \{v : \langle v, w\rangle = 0 \text{ for all } w \in V\}$, then there is an orthogonal basis for V consisting of eigenvectors of A.

An element $A \in \mathfrak{a}$ is *regular* if $\lambda(A) \neq 0$ for all $\lambda \in \Sigma$. Then the set of regular elements is the complement of a finite union of hyperplanes, and hence is dense. If $A \in \mathfrak{a}$ is regular, it follows that $[A, \mathfrak{k}] = \mathfrak{p} \ominus \mathfrak{a}$. This establishes (16). ∎

A similar result may be derived with $\mathfrak{k}$ playing the role of $\mathfrak{p}$.

THEOREM **9** *Let $\mathfrak{k}$ be a compact Lie algebra, let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$, and let K denote any compact Lie group with Lie algebra $\mathfrak{k}$. Then an $Ad(K)$-invariant function on $\mathfrak{k}$ is convex on $\mathfrak{t}$ if and only if it is convex on $\mathfrak{k}$.*

The basic example to keep in mind is $\mathfrak{k}$ the skew-hermitian $d \times d$ matrices, K the unitary group on $\mathbf{C}^d$, and $\mathfrak{t}$ the diagonal matrices in $\mathfrak{k}$. This is the complex version of the Chandler Davis result. The real Davis result may be derived as a special case of the complex result as follows.

Let F be a rotation-invariant function on symmetric matrices, and let f be its convex restriction to the diagonals. Let $g(x) = f(\Im x)$; then g is convex on the set of diagonal skew-hermitian matrices hence extends to a unitarily-invariant convex function G on $\mathfrak{k}$. For x real symmetric, let $\tilde{F}(x) = G(ix)$; then $\tilde{F}$ is rotation invariant, convex, and extends f off the diagonals. Hence $\tilde{F} = F$, establishing the convexity of F.

For another application, let $\mathfrak{k} = \mathfrak{so}(\mathbf{R}^d)$ with d even, $K = SO(\mathbf{R}^d)$, and $\mathfrak{t}$ the space of diagonal[3] matrices; an example of a non-trivial convex Ad-invariant function on $\mathfrak{k}$ is the reciprocal of the Pfaffian [18]; this is best explained in terms of anti-commuting calculus [3].

Let $x_1, \dots, x_d$ be anti-commuting variables $x_i x_j + x_j x_i = 0$, $i, j = 1, \dots, d$. If $y_i = \sum_{j=1}^d a_{ij} x_j$ is a linear transformation, then we have the elementary

$$y_1 y_2 \dots y_d = \det(A) x_1 x_2 \dots x_d. \tag{17}$$

Below, by a function f of $x_1, \dots, x_d$, we mean any element of the algebra generated by $x_1, \dots, x_d$. Given such a function, its Berezin integral [3] $\int f(x_1, \dots, x_d)\, dx_1 \dots dx_d$ is by definition the coefficient of $x_1 x_2 \dots x_d$ in f. With this interpretation, (17) may be re-written as the change-of-variable formula

$$\int f(y_1, \dots, y_d)\, dx_1 \dots dx_d = \det(A) \int f(y_1, \dots, y_d)\, dy_1 \dots dy_d$$

(note the determinant is on the "wrong" side).

Given $A \in \mathfrak{so}(\mathbf{R}^d)$, its Pfaffian is the anti-commuting Gaussian integral [3]

$$\mathrm{Pf}(A) = \int \exp\left(\frac{1}{2} \sum_{i,j=1}^{d} a_{ij} x_i x_j\right) dx_1 \dots dx_d.$$

Here by exp we mean the exponential series which terminates after finitely many terms because of anti-commutativity. By the change-of-variable formula, this exhibits the Pfaffian as a rotationally invariant polynomial function of the entries of A.

[3]Here "diagonal" means consisting of 2×2 skew-symmetric blocks along the diagonal.

Now let $x_1,\dots,x_d,y_1,\dots,y_d$ be anti-commuting variables and set $z_i = x_i + \sqrt{-1}y_i$, $\bar{z}_i = x_i - \sqrt{-1}y_i$, $i = 1,\dots,d$. Then it is straightforward to check the complex Gaussian formula

$$\int \exp\left(\sum_{i,j=1}^{d} a_{ij} z_i \bar{z}_j\right) (dz_1 d\bar{z}_1)\dots(dz_d d\bar{z}_d) = \det(A),$$

valid for any real A (here the integral equals the coefficient of $\prod_{i=1}^{d} z_i\bar{z}_i$). When A is skew-symmetric, substituting the expressions for $z_i, \bar{z}_j$ yields the well-known fact

$$(\mathrm{Pf}(A))^2 = \det(A), \qquad A + A^t = 0.$$

When A is diagonal, $A \in \mathfrak{so}(\mathbf{R}^d)$, A is a sum of 2×2 skew-symmetric blocks $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$. In this case $\mathrm{Pf}(A)$ reduces to the product of the top right entries a. Say $A \in \mathfrak{so}(\mathbf{R}^d)$ is *positive* if these top right entries are positive. Defining $\log : \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$ by $\log x = -\infty$ for $x \le 0$, we see that the set of positive skew-symmetric matrices is convex and that Pf is log-concave on it.

Now we turn to the proof of Theorem 9, which is similar to that of the previous one.

PROOF: Since $\mathrm{ad}(\mathfrak{t})$ is a commuting set of skew-adjoint operators, they can be simultaneously diagonalized into 2×2 blocks. This leads to a finite set Δ of linear functionals α in $\mathfrak{t}^*$ and nonzero vectors $X_\alpha, Y_\alpha \in \mathfrak{k}$, such that with $\mathfrak{k}_\alpha = \mathbf{R}X_\alpha + \mathbf{R}Y_\alpha$, we have

$$[H, X_\alpha] = \alpha(H)Y_\alpha \text{ and } [H, Y_\alpha] = -\alpha(H)X_\alpha, \qquad H \in \mathfrak{t}, \alpha \in \Delta$$

and

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{k}_\alpha$$

is an orthogonal direct sum.

An element $H \in \mathfrak{t}$ is *regular* if $\alpha(H) \neq 0$ for all $\alpha \in \Delta$. Then the set of regular elements is the complement of a finite union of hyperplanes, and hence is dense. If $H \in \mathfrak{t}$ is regular, it follows that $[H, \mathfrak{k}] = \mathfrak{k} \ominus \mathfrak{t}$. This establishes (16). ∎

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