Modern calendar and continued fractions

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1 Very brief history

There are many different calendars on earth in use today: Hebrew, Chinese, Hindu, Ethiopian etc. These various calendar systems may seem completely artificial, a pure product of human arbitrary choice, like a language. Yet, the basis of all calendars are the observed periodic motion of the Sun and the Moon across the skies. Here's the numbers. One year is 365.24219878 days and one lunar month is 29.530589 days long. The ratio of these two numbers is 12.368267 and it it is equal to the number of lunations in a year. So, you see that the lunar month is between 29 and 30 days long, while there is about 12 lunar months a year. The oldest Babylonian calendar was a lunar one of 12 months consisting alternately of 29 and 30 days in accordance with these numbers. Observe that the calendar year in such lunar calendar contains just 354 days, a gross underestimate. But by at least the 5th millennium BC this calendar was replaced by an Egyptian calendar of 12 months, each consisting of 30 days.

The Egyptian calendar had only 360 days in a year and the discrepancy was soon noticed. To adjust the calendar, five days, the epagomenes, were added at the end of the 360-day year in Pharaonic times. This 365 day calendar was in effect for more than 3000 years of Pharaohs until 238 B.C. In a remarkable Decree of Canopus by Ptolemy III, a sixth epagomenal day was introduced every fourth year. This is so called Alexandrian calendar. It survives nowadays in the calendars of Coptic and Ethiopian churches.

Our calendar is a direct descendant of the ancient Roman calendar. Up until 46 B.C. Romans used a 365 day year. During his Egyptian campaign Julius Caesar learned about the Alexandrian calendar with its 4-year leap year cycle, that was much more precise than the current Roman calendar of 365 days. Along with him Caesar brought the Alexandrian astronomer Sosigenes, upon whose advice he based his calendar reform, creating the Julian calendar. The mean year length for Julian calendar is 365.25 days, which is very close to the more precise number 365.24219878. The Julian calendar was so good that it accumulated only one day error in about a hundred years. Yet, over the next millennium, the discrepancy was noticed and suggestions were made to correct it. Finally, in 1582 Pope Gregory XIII assembled a commission to design a new more precise calendar system. The main author of the new system was the Naples astronomer Aloysius Lilius. Following the recommendation of his commission, Pope Gregory XIII decreed that the day following Oct. 4, 1582 would be Oct. 15; that the years ending in "00" would be common years rather than leap years - except those divisible by 400 and that New Year will start on January 1. The non-Catholic world perceived the calendar decree as a Catholic ploy. It took nearly 200 years for the change to come about. Great Britain and her colonies made the change in 1752 when September 2nd was followed by September 14 and New Year's Day was changed from March 25 to January 1.

If your computer has a calendar program that can display calendars for 1582 and 1752, you can check the religious faith of your computer. For example, on my Linux system the results are

```
    October 1582                 September 1752

Su Mo Tu We Th Fr Sa          Su Mo Tu We Th Fr Sa
    1  2  3  4  5  6                 1  2 14 15 16
 7  8  9 10 11 12 13          17 18 19 20 21 22 23
14 15 16 17 18 19 20          24 25 26 27 28 29 30
21 22 23 24 25 26 27
28 29 30 31
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making my computer a protestant.

The year 2000 is one of the rare leap years that end in "00". The next time this happens will be 400 years from now. The Gregorian calendar is both precise (1 day error in about 3,300 years) and convenient. Is it an art to come up with such a design or is there a science behind it? Continued fractions provide just such a science.

2 Continued fractions

The history of continued fractions can be traced back to an algorithm of Euclid. Let us recall this algorithm. Suppose we would like to find the greatest common divisor of numbers 75 and 33.

$$75 = 2 \cdot 33 + 9 \qquad \frac{75}{33} = 2 + \frac{9}{33}$$

$$33 = 3 \cdot 9 + 6 \qquad \frac{75}{33} = 2 + \frac{9}{3 \cdot 9 + 6} = 2 + \cfrac{1}{3 + \cfrac{6}{9}}$$

$$9 = 1 \cdot 6 + 3 \qquad \frac{75}{33} = 2 + \cfrac{1}{3 + \cfrac{6}{1 \cdot 6 + 3}} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{3}{6}}}$$

$$6 = 2 \cdot 3 \qquad \frac{75}{33} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{3}{2 \cdot 3}}} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

The last non-zero remainder, 3 in our case, is the greatest common divisor of 75 and 33. There is no evidence though that Greeks knew about the connection between the left column and the right column above. The first continued fraction was used in 1572 by Bombelli to approximate $\sqrt{13}$. The first infinite continued fraction appears in 1659 in the work of Lord Brouncker to expand $4/\pi$. It is Euler's systematic development of the theory starting in 1737 that showed the value of the notion for both number theory and analysis. A torrent of results followed. In 18th and 19th centuries everybody who was anybody in mathematics contributed. If the number is rational the continued fraction terminates like for 75/33. If the number is irrational the continued fraction goes on forever. For example, for the irrational number $\sqrt{2}$ we can execute the Euclidean algorithm, in essence looking for the greatest common divisor of $\sqrt{2}$ and 1. The algorithm will never terminate since the two numbers are incommensurate.

$$\sqrt{2} = 1 + 0.41421356\ldots = 1 + \frac{1}{2.41421356\ldots} =$$

$$1 + \frac{1}{2 + 0.41421356\ldots} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + 0.41421356\ldots}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + 0.41421356\ldots}}} = \ldots$$

concluding

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

The esthetic beauty of continued fractions may go some ways towards justifying the significance of some numbers from algebra or geometry. The continued fraction expansion

$$\tau = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

would suggest that the number $\tau = (1+\sqrt{5})/2$ has some significance. In fact, this number is none other than the "golden ratio".

If we terminate the infinite continued fraction for the irrational number α at the nth step we will obtain a rational approximation α_n to α. The rational number α_n is called the nth convergent for α. For example, the first 4 convergents to numbers $\sqrt{2}$ and π are

$$\alpha = \sqrt{2} = 2.41421356\ldots \qquad \pi = 3.141592654\ldots$$

$$\alpha_0 = 1 \qquad \pi_0 = 3$$

$$\alpha_1 = \frac{3}{2} = 1 + \frac{1}{2} \qquad \pi_1 = \frac{22}{7} = 3 + \frac{1}{7}$$

$$\alpha_2 = \frac{7}{5} = 1 + \cfrac{1}{2 + \cfrac{1}{2}} \qquad \pi_2 = \frac{333}{106} = 3 + \cfrac{1}{7 + \cfrac{1}{15}}$$

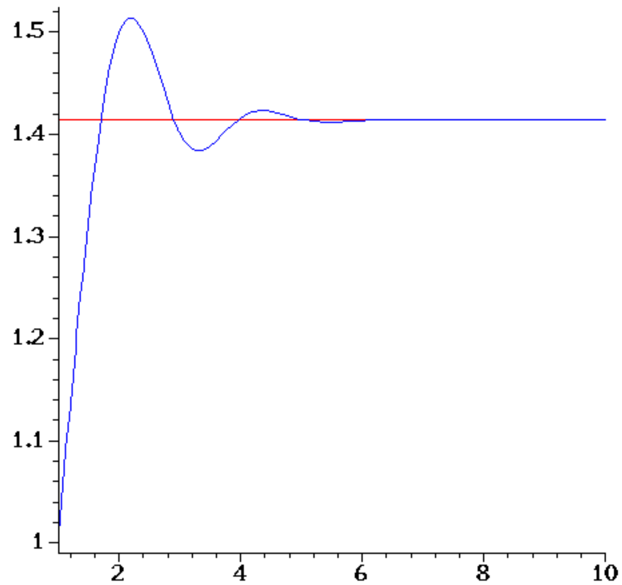
$$\alpha_3 = \frac{17}{12} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} \qquad \pi_3 = \frac{355}{113} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

$$\alpha_4 = \frac{41}{29} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} \qquad \pi_4 = \frac{103993}{33102} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

The name convergent comes from the fact that convergents do converge to the number. For example,

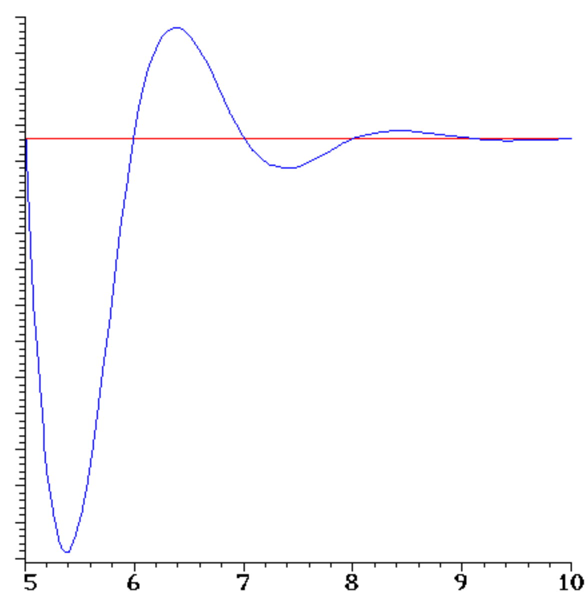
$$\alpha - \alpha_4 \approx 4.2 \times 10^{-4} \qquad \pi - \pi_4 \approx 5.8 \times 10^{-10}$$

Here is the graph for $\sqrt{2}$



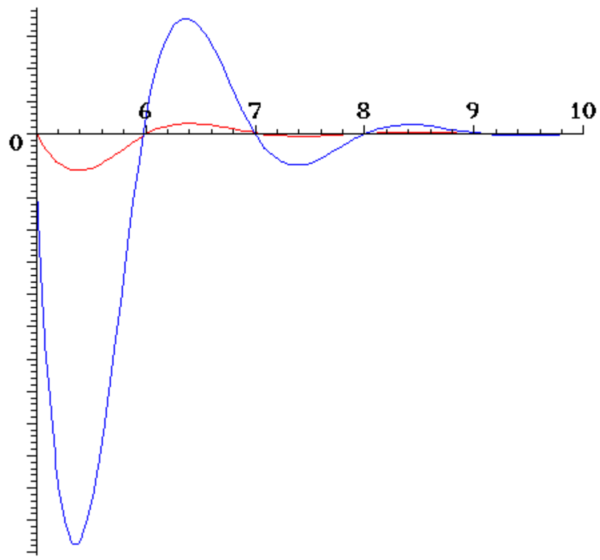
We see that convergents alternately lie above and below the exact value of $\sqrt{2}$.

Here is the graph for π.



We see the same alternating pattern of approximation. In fact, this is true in general for any number.

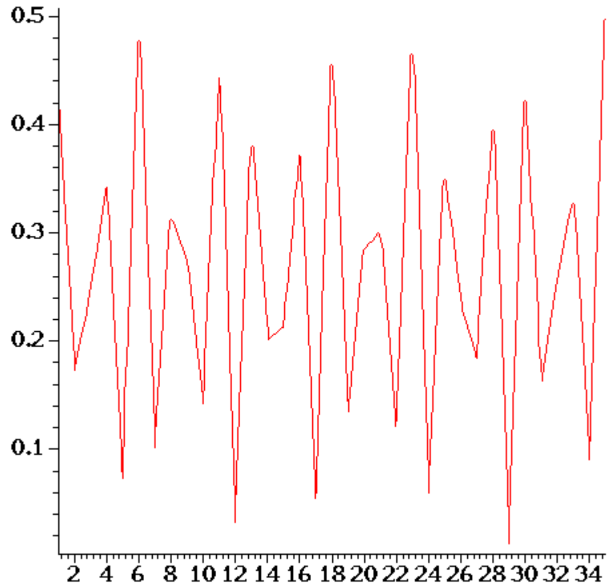
The speed of convergence of continued fractions to a number they represent varies from number to number (but it is always very very fast). Here is a comparison between the convergence errors for $\sqrt{2}$ (blue) and π (red).



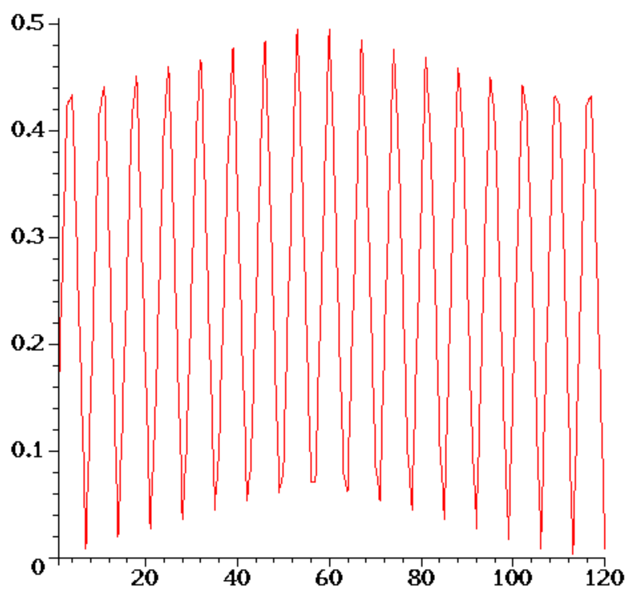
The continued fraction expansions have many remarkable properties. We will be interested mainly in its approximating power relevant for the design of a good calendar system. It turns out that the convergents α_n for the irrational number α have superior approximating properties. The following definition makes it precise what we mean by a good approximation.

Definition 1 *The fraction p/q is called a good approximation for α if for any $q' < q$ and any integer p' we have*

$$|q\alpha - p| < |q'\alpha - p'|$$



The good approximations for $\sqrt{2}$ occur when q=2, 5, 12 and 29. The next good approximation occurs when q=70.



The good approximations for π occur at q=7, 106 and 113. The next good approximation does not occur before q=33,102.

Observe that the numbers q are exactly the denominators in the convergents for $\sqrt{2}$ and π respectively. This is not an accident and holds in general for all convergents and for all numbers α. We state it precisely and unambiguously in the form of a Theorem.

THEOREM 1 *Every convergent α_n is a good approximation (in the sense of Definition 1) for α and conversely, every good approximation for α is one of the numbers α_n for some $n \geq 1$. In fact q_n is the smallest integer $q > q_{n-1}$ such that*

$$|q\alpha - p| < |q_{n-1}\alpha - p_{n-1}|$$

for some integer p.

We also have the inequalities

$$\frac{1}{2q_{n+1}} < |q_n\alpha - p_n| \leq \frac{1}{q_{n+1}}.$$

The proof of the theorem is given in the book of Serge Lang. It is not very difficult to follow Lang's proof but quite tricky to discover it on your own. Christian Huygens was the first to describe the sense in which continued fractions give the best approximations of real numbers.

Now that you know that continued fractions are very good at approximating numbers rational and irrational, it is not surprising to find them in many unusual (at first glance) places. Looking deeper at continued fractions you would discover many amazing properties of these objects. We can say that there is music in continued fractions. Speaking of music, there are also continued fractions in music. Armed with continued fractions we return to the calendar and discover how continued fractions can explain more or less any calendar system that was ever proposed or implemented.

3 Calendar and continued fractions

The idea of a modern calendar is to have a cycle spanning q years p of which are leap years while the remaining $q - p$ years are not. The numbers p and q should be chosen so that the mean year length is as close to the astronomical year as possible. In addition, the cycle length q and the rule for selecting p leap years should be easy to use and convenient to implement. The Julian 4 year cycle as well as Gregorian 400 year cycle are examples of such easy and convenient calendar systems. By contrast, the Hebrew 19 year cycle requires a calculator to figure out the leap years (where an extra month, not day, is added).

Consider now the cycle of q years during which there are p leap years. During the cycle $365q + p$ days pass. This makes the mean year length to be $365 + p/q$ days. Then we need to find a "convenient" value for q that makes p/q as close to $\alpha = 0.24219878$ as possible. We already know that we need to examine the sequence of convergents coming from the continued fraction expansion of the number α. For $\alpha = 0.24219878$ we have

$$0.24219878 = \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \ddots}}}}}$$

which gives the following sequence of convergents:

$$\frac{p_1}{q_1} = \frac{1}{4}, \ \frac{p_2}{q_2} = \frac{7}{29}, \ \frac{p_3}{q_3} = \frac{8}{33}, \ \frac{p_4}{q_4} = \frac{31}{128}, \ \frac{p_5}{q_5} = \frac{163}{673}.$$

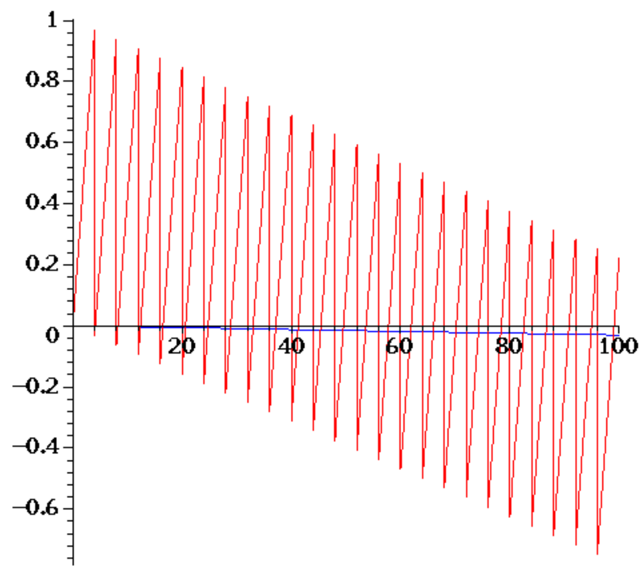
The first fraction in the sequence corresponds to the Julian 4 year cycle system with a single leap year in the cycle. The remaining fractions offer very inconvenient cycle lengths: 29, 33, 128 and 673 years respectively. They are, therefore, rejected. (Nevertheless, the idea of a 33-year period has crossed people's minds. Such a calendar would indeed be more precise than the current Gregorian calendar, but less precise than the 500-year cycle calendar discussed below.) Instead, we would rather have a cycle several centuries long, if the leap year selection rule is simple enough. So, assume that $q = 100q'$, where q' must be an integer between 1 and 9. This corresponds to the problem of approximating the number $\alpha' = 100\alpha = 24.219878$ by rationals.

$$0.24219878 \times 100 = 24 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}$$

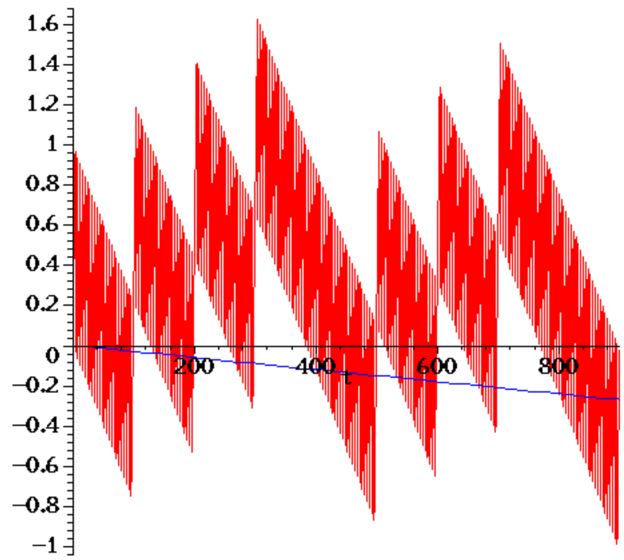
We easily compute first 4 convergents:

$$\frac{p_1}{q_1} = \frac{97}{4}, \ \frac{p_2}{q_2} = \frac{121}{5}, \ \frac{p_3}{q_3} = \frac{218}{9}, \ \frac{p_4}{q_4} = \frac{993}{41}.$$

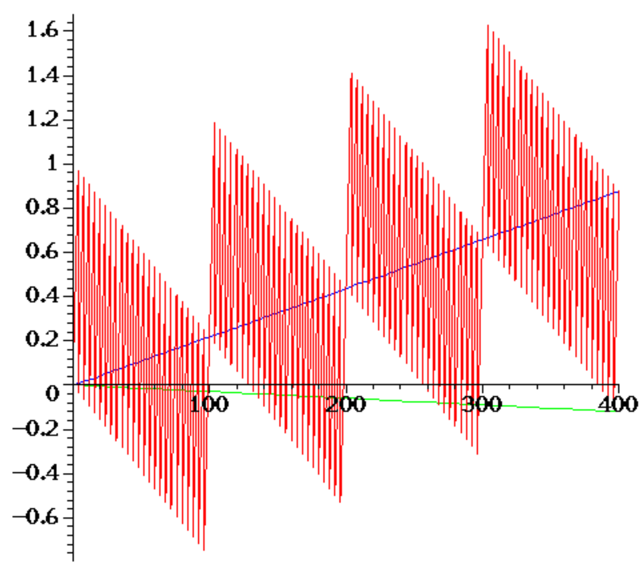
We see that we have three candidates for the calendar model. The first one corresponds to our Gregorian calendar. It is based on a 400 year cycle with 97 leap years: all those divisible by 4 (there is a hundred of them) except 100th, 200th and 300th years making up the needed 97 leap years in a cycle. The next fraction 121/5 corresponds to a 500 year cycle calendar with 121 leap years in each cycle. In such a calendar every year divisible by 4 would be a leap year unless it is divisible by 100 with the exception of years divisible by 500, which are still leap years. This system is as simple and as convenient as the Gregorian calendar and provides a better accuracy. The Gregorian year is 26 seconds longer than the solar year resulting in 1 day error each 3,320 years. The 500 year cycle calendar is 17 seconds shorter than the solar year resulting in 1 day error each 5,031 years. The Pope missed that one. The last choice for the calendar offers a 900 year cycle. However, with 218 leap years in the cycle the calendar requires to make 7 exceptions to the fourth year leap rule ($218 = 900 \div 4 - 7$). Making this arrangement would create a more complicated calendar. And besides, the 900 year cycle may be just a bit too long to be convenient. So, we would reject this more precise calendar in favor of the simpler ones.



This graph shows the difference between our Gregorian calendar time and the true solar time over 100 years. The sawtooth oscillations are the insertions of leap years every four years.

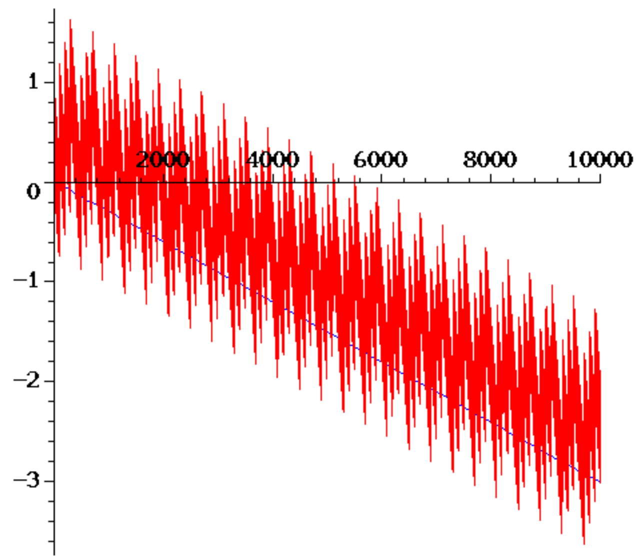


This graph shows the difference between our Gregorian calendar time and the true solar time over 900 years. The individual leap year insertions are almost invisible. We clearly see the effect of leap year omissions every century and the effect of the leap year every 4 centuries. In fact, if we omit the 400 year rule but keep omitting leap years every century the calendar error will look like this:



The green line shows the Gregorian calendar error for comparison.

Even Gregorian calendar will accumulate a large error. Eventually.



Here the individual leap years are no longer visible. The smaller oscillations are centennial omissions of leap years. These are grouped into repeating packets of four. We see that our calendar accumulates error at the rate of about 1 day every 3,300 years.

We might speculate what can be done in the future to correct for the slowly accumulating error of the Gregorian calendar. The idea is to keep the old system but make some very infrequent corrections. Continued fractions come handy here again. In other words we are looking for a much longer cycle length q, which would comprise several 400 year cycles $q = 400q'$, where q' is the number of 400 year cycles in the new longer cycle. We then expand 400×0.24219878 into a continued fraction.

$$400 \times 0.24219878 = 96 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{3 + \cfrac{1}{2 + \ddots}}}}$$

Convergents are $96, \quad 97, \quad \frac{775}{8}, \quad \frac{2422}{25}, \quad \frac{5619}{58}, \quad \ldots$ The third convergent suggests a $8 \times 400 = 3,200$ year cycle with 775 leap years altogether. Recall, that according to the Gregorian calendar, there is 97 leap years in each 400 year cycle. So, within 8 cycles we will have $8 \times 97 = 776$ leap years. Thus, canceling the leap year every 3,200 years will allow us to keep Gregorian calendar in the intervening time, while making it much more precise. The new system would accumulate a 1 day error in 100,000 years, that is never.

An even more interesting scenario would have been possible had the Pope done his math. If our calendar was based on a 500 year cycle suggested above, then we would be expanding 500×0.24219878 into a continued fraction.

$$500 \times 0.24219878 = 121 + \cfrac{1}{10 + \cfrac{1}{16 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}}$$

with convergents

$$[121, \frac{1211}{10}, \frac{19497}{161}, \frac{59702}{493}, \frac{138901}{1147}, \frac{337504}{2787}, \ldots].$$

The second convergent 1211/10 suggests a new cycle length of 5,000 years with 1211 leap years in the cycle. The 500 year cycle calendar would have 1210 leap years in 5,000 years. In order to make 1211 leap years we might want to have February 30, 5000 in celebration of the 5th millennium. The 5,000 year cycle calendar will accumulate a 1 day error in a whopping 1 million years. This system has been suggested by Bernard Rasof (”Continued fractions and 'leap' years”, *The Mathematics Teacher*, **63**, pp. 144-148, 445, 1970.) Be it as it may, either the Pope didn't do his math (which I find unlikely), or the astronomical data was not precise enough at the time to justify the 500 year cycle, or he had other reasons for settling on the current calendar (for example, the coming-soon 1600 would not increase the discrepancy between the two versions of the calendar under the 400 year cycle).

The continued fractions can also be used to discover the 19 year Metonic cycle of the Hebrew calendar. In lunar calendars an extra month (from new moon to new moon) is inserted in a leap year. As we mentioned in the beginning, there is 12.368267 lunations a year. Expanding this number into a continued fraction we obtain

$$12.368267 = 12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{17 + \ddots}}}}}}$$

with convergents $12, \frac{25}{2}, \frac{37}{3}, \frac{99}{8}, \frac{136}{11}, \frac{235}{19}, \frac{4131}{334}, \ldots$. The Metonic cycle corresponds to the sixth convergent $\frac{235}{19}$, meaning that there is approximately 235 lunations in 19 years.

If all years contained 12 months then in 19 years we would have $19 \times 12 = 228$ months. Therefore, we need to insert 7 more months to make it to 235. The actual leap year rule requires a calculator: The year Y is a leap year if $7Y + 1 \pmod{19} < 7$.

If you want to learn more about continued fractions the books

- Lang, Serge *Introduction to Diophantine approximations.* Second edition. Springer-Verlag, New York, 1995.
- Jones, William B.; Thron, Wolfgang J. *Continued fractions. Analytic theory and applications.* With a foreword by Felix E. Browder. With an introduction by Peter Henrici. Encyclopedia of Mathematics and its Applications, 11. Addison-Wesley Publishing Co., Reading, Mass., 1980.

are excellent references. The calendar history and continued fractions are also discussed in two Mathematical Intelligencer articles:

- Dutka, Jacques "On the Gregorian revision of the Julian calendar", *Math. Intelligencer*, **10** (1988), no. 1, 56–64.
- Rickey, V. Frederick "Mathematics of the Gregorian calendar", *Math. Intelligencer*, **7** (1985), no. 1, 53–56.

There is another web site that discusses both the calendar and the continued fractions. It focuses, however, more on the calendar part than on continued fractions.

Finally, I would like to mention that I got my idea for doing the public lecture about it on February 29, 2000 from an article in January/February 2000 issue of one of my favorite (and no longer published in the US) magazines Quantum.