Complete characterization of symmetric Kubo-Ando operator means satisfying Molnár's weak associativity

Yury Grabovsky*
Temple University
Department of Mathematics
Philadelphia, PA, USA 19122

Graeme W. Milton†
University of Utah
Department of Mathematics
Salt Lake City, UT, USA 84112

Aaron Welters‡
Florida Institute of Technology
Department of Mathematics and Systems Engineering
Melbourne, FL, USA 32901

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Abstract

We provide a complete characterization of a subclass of weakly associative means of positive operators in the class of symmetric Kubo-Ando means. This class, which includes the geometric mean, was first introduced and studied in L. Molnár, *Characterizations of certain means of positive operators*, Linear Algebra Appl. 567 (2019) 143-166, where he gives a characterization of this subclass (which we call the Molnár class of means) in terms of the properties of their representing operator monotone functions. Molnár's paper leaves open the problem of determining if the geometric mean is the only such mean in that subclass. Here we give a negative answer to this question by constructing an order-preserving bijection between this class and a class of real measurable odd periodic functions bounded in absolute value by 1/2. Each member of the latter class defines a Molnár mean by an explicit exponential-integral representation. From this we are able to understand the order structure

*Email: yury@temple.edu
†Email: graeme.milton@utah.edu
‡Email: awelters@fit.edu

of the Molnár class and construct several infinite families of explicit examples of Molnár means that are not the geometric mean. Our analysis also shows how to modify Molnár's original characterization so that the geometric mean is the only one satisfying the requisite set of properties.

Keywords: Arithmetic-harmonic-geometric means, Kubo-Ando means, operator monotone functions, Herglotz-Nevanlinna functions, Molnár class of means

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1 Introduction

In this paper, following [24], we adopt the following notations:

- H always denotes a complex Hilbert space.
- $B(H)$ - the set of all bounded linear operators on H.
- $B(H)^+$ - the set of all positive semidefinite operators in $B(H)$.
- $B(H)^{++}$ - the set of all invertible operators in $B(H)^+$.
- I denotes the identity operator on H.
- The order relation $A \leq B$ for $A, B \in B(H)^+$ means $B - A \in B(H)^+$ [i.e., $\leq$ is the Loewner order on $B(H)^+$].
- We write $A_n \downarrow A$ if (A_n) is a monotonically decreasing sequence in $B(H)^+$, i.e., $A_1 \geq A_2 \geq \ldots$, and A_n converges strongly to A in $B(H)$.

Following [17] (see also [14] and [29, Chap. 36 and 37]), a binary operation

$$\sigma : B(H)^+ \times B(H)^+ \to B(H)^+,\ (A, B) \mapsto \sigma(A, B) =: A\sigma B$$

is called a *connection*[1] on $B(H)^+$ if the following requirements are fulfilled:

(I) (Joint Monotonicity) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$;

(II) (Transformer Inequality) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$;

(III) (Upper Semicontinuity) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$.

Moreover, a connection σ is called a *mean* (or, more precisely, a *Kubo-Ando mean*) or *symmetric*, respectively, if

(IV) (Normalization) $I\sigma I = I$;

(V) (Permutation Symmetry) $A\sigma B = B\sigma A,\ \forall A, B \in B(H)^+$.

[1]The term comes from the study of connections of elements in an electrical network [3, 4].

In particular, a *symmetric Kubo-Ando mean* is any binary operation σ on $B(H)^+$ satisfying (I)-(V).

Three important examples of symmetric Kubo-Ando means (which motivated the axiomatic approach of [17] to positive operator means) are the *arithmetic mean* ∇, *harmonic mean* !, and *geometric mean* # (see also [27, 18, 5] and the recent survey [19]) which, for all $A, B \in B(H)^{++}$, are given by:

$$A \nabla B = \frac{1}{2}(A + B); \tag{1.1}$$

$$A!B = 2\left(A^{-1} + B^{-1}\right)^{-1}; \tag{1.2}$$

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}. \tag{1.3}$$

The values of these means at non-invertible elements of $B(H)^+$ can be determined from (III) by passing to the limit of monotonically decreasing sequences from $B(H)^{++}$ (see, e.g., [17]). For instance, if $A, B \in B(H)^+$ then the formula for $A\nabla B$ is still (1.1), whereas if $A \in B(H)^{++}$, $B \in B(H)^+$ then the formula for $A\#B$ is still (1.3). But in general, if $A, B \in B(H)^+$ then (1.2) and (1.3) become

$$A!B = 2 \lim_{n\to\infty} \left(A_n^{-1} + B_n^{-1}\right)^{-1}, \tag{1.4}$$

$$A\#B = \lim_{n\to\infty} A_n^{1/2}(A_n^{-1/2}B_nA_n^{-1/2})^{1/2}A_n^{1/2},$$

for any two sequences $(A_n), (B_n)$ in $B(H)^{++}$ such that $A_n \downarrow A$ and $B_n \downarrow B$.

Aside from the factor of 2, the right hand side of (1.4) has been given the name 'parallel sum.' More precisely, the parallel sum is denoted by : and gives another well-known example of a connection:

$$A : B = \frac{1}{2}A!B, \text{ for } A, B \in B(H)^+. \tag{1.5}$$

The concept of the parallel sum was introduced for positive semidefinite matrices in [3] and extended to bounded (positive semidefinite) linear operators in [4] (see also [17, 25]).

As shown in [17], there is an important relationship between connections, Loewner's theory on operator-monotone functions, and the properties of a special class of analytic functions called Herglotz functions [13, 12, 30] (also called Nevanlinna [2, 21], Herglotz-Nevanlinna [6, 22, 23, 26], Pick [7, 8, 11, 9], or R-functions [31, 32, 33, 15], [16, Appendix]). Let us briefly elaborate on this.

First, a function f is called a *Herglotz function* if $f : \mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R}$ is analytic, where $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denotes the open upper half-plane. Next, let us introduce the following notation.

Notation 1. *Let OM_+ denote*[2] *the class of all analytic functions $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ satisfying $f(x) \geq 0$ if $x \in \mathbb{R}$ with $x > 0$ and $\operatorname{Im} f(z) \geq 0$ if $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$.*

[2] The restrictions of functions in $OM_+ \setminus \{0\}$ to the complex upper half-plane is denoted by $\mathscr{S}([-\infty, 0])$ in [16, Appendix, Sec. 4]. The functions in this class, and therefore, in $OM_+ \setminus \{0\}$, have an exponential-integral representation [16, Appendix, Sec. 5] that will play an important role in our paper.

In particular, if $f \in OM_+$ then its restriction $f|_{\mathbb{C}^+} : \mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R}$ to $\mathbb{C}^+$ is a Herglotz function. Next, a function

$$f : (0, \infty) \to [0, \infty)$$

is called a *(positive) operator monotone function* if, for every Hilbert space H,

$$A, B \in B(H)^{++},\ A \leq B \Rightarrow f(A) \leq f(B),$$

where f is defined on $B(H)^+$ by the functional calculus for self-adjoint operators on H [28, 30]. A deep result of Loewner [20] (see also [11, 29]) says that every positive operator monotone function g has a unique analytic continuation to a function $f \in OM_+$ [or, equivalently, $f(x) = g(x)$ for all $x > 0$] and conversely. *Given this correspondence between positive operator monotone functions and the elements of OM_+, we will abuse notation throughout the rest of this paper and not distinguish them unless necessary.*

More precisely, the map $f \mapsto f|_{(0,\infty)}$ sending $f \in OM_+$ to its restriction $f|_{(0,\infty)} : (0, \infty) \to [0, \infty)$ is a bijection of OM_+ onto the class of positive operator monotone functions. Furthermore, the map $m \mapsto f$, defined by

$$f(z) = a + bz + \int_{(0,\infty)} \frac{z(1+\lambda)}{z+\lambda} dm(\lambda),\ \text{for } z \in \mathbb{C} \setminus (-\infty, 0], \tag{1.6}$$

where $a = m(\{0\})$ and $b = m(\{\infty\})$, establishes a bijection between the class of finite (positive) Borel measures on $[0, \infty]$ onto this class OM_+ of functions.

Now a result of [17] says that positive operator monotone functions are in a bijective correspondence with connections, whereby a positive operator monotone function f gives rise to a connection $\sigma = \sigma_f$ on $B(H)^+$ by the formula

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

for all $B \in B(H)^+, A \in B(H)^{++}$ which is extended uniquely to all $B(H)^+$ by property (III). Conversely, function f can be recovered from a connection σ by the formula

$$f(x)I = I\sigma(xI),\ \text{for } x > 0. \tag{1.7}$$

The function f is called the *representing function* of σ. Furthermore, if H is infinite dimensional then by a deep result of [17], each and every connection σ on $B(H)^+$ arises in this way, i.e., $\sigma = \sigma_f$ for a unique operator monotone function f. Moreover, by [17, Theorem 3.4] (see also [17, Lemma 3.1]) it is known that, in terms of the representation (1.6),

$$A\sigma B = aA + bB + \int_{(0,\infty)} \frac{1+\lambda}{\lambda} [(\lambda A) : B] dm(\lambda),\ \text{for } A, B \in B(H)^+,$$

where $a = m(\{0\}), b = m(\{\infty\})$, : denotes the parallel sum as def. by (1.5), and this establishes a bijection, $m \mapsto \sigma$, between the class of finite (positive) Borel measures on $[0, \infty]$ onto the class of connections.

For instance, by (1.7) and [17, Corollary 4.2], it is known that a representing function f corresponding to a connection σ that is a mean or symmetric is one which satisfies, respectively,

$$f(1)=1; \quad f(x)=xf\left(\frac{1}{x}\right), \text{ for } x>0.$$

For example, the representing functions $f_{\nabla}, f_{!}, f_{\#}$ of the arithmetic mean ∇, harmonic mean !, and geometric mean $\#$, respectively, are given by

$$f_{\nabla}(x)=\frac{1}{2}(1+x); \tag{1.8}$$

$$f_{!}(x)=\frac{2x}{1+x}; \tag{1.9}$$

$$f_{\#}(x)=\sqrt{x}. \tag{1.10}$$

A natural question arises that was considered by Kubo-Ando [17, Sec. 4 and 5] and, more recently, by L. Molnár [24]: What properties of a symmetric Kubo-Ando mean $\sigma=\sigma_f$ characterize it as the arithmetic mean ∇, harmonic mean !, or geometric mean $\#$? Equivalently, in terms of representing functions f (or, equivalently, on operator monotone functions f), what are necessary and sufficient conditions that guarantee $f\in\{f_{\nabla}, f_{!}, f_{\#}\}$?

To address this question, L. Molnár [24] considered an algebraic characterization of those means in terms of a weak form of an associativity law of a binary operation on $B(H)^{++}$. His main results in this regard can be summarized as follows (see [24, Theorems 6 and 8]).

Theorem 2 (L. Molnár)**.** *Let H be a complex Hilbert space with $\dim H\geq 2$ and σ be a symmetric Kubo-Ando mean on $B(H)^{++}$ with representing operator monotone function f. Assume that there exists a continuous strictly increasing and surjective function $g:(0,\infty)\to(0,\infty)$ such that the operation*

$$\diamond:(A,B)\mapsto g(A\sigma B), \text{ for } A,B\in B(H)^{++}$$

is either associative, i.e.,

$$(A\diamond C)\diamond B=A\diamond(C\diamond B), \ \forall A,B,C\in B(H)^{++}, \tag{1.11}$$

or satisfies the weaker form of associativity

$$(A\diamond I)\diamond B=A\diamond(I\diamond B), \ \forall A,B\in B(H)^{++}. \tag{1.12}$$

If (1.11) is satisfied then σ is the arithmetic or harmonic mean. On the other hand, if it satisfies (1.12) then either we have $g(f(x))=x$, $x>0$ [meaning that $A\diamond I=I\diamond A=A$, $A\in B(H)^{++}$] or we have one of the following three possibilities:

(a) there is a positive scalar $c\neq 1$ such that $f(c^2x)=cf(x)$, for $x>0$;

(b) σ is the arithmetic mean;

(c) σ is the harmonic mean.

He also proves (see [24, pp. 160-161]) the following partial converse of this theorem.

Theorem 3 (L. Molnár)**.** *If σ is a symmetric Kubo-Ando mean with representing operator monotone function f such that (a), (b), or (c) is true in Theorem 2 then there is a continuous strictly increasing and surjective function $g : (0,\infty) \to (0,\infty)$ with $g \neq f^{-1}$ such that the operation $\diamond : (A,B) \mapsto g(A\sigma B)$, $A, B \in B(H)^{++}$ satisfies (1.12).*

This motivates the following definition of a special class of positive operator means considered by L. Molnár.

Definition 4 (Molnár mean)**.** *A symmetric Kubo-Ando mean σ is called a Molnár mean if its representing function f has property (a) in Theorem 2. The set of all Molnár means will be called the Molnár class of means.*

Notice that the geometric mean $\#$ is a Molnár mean with representing function $f_{\#}$. Because of this, L. Molnár [24, p. 161] posed the following problem, which we have rephrased in terms of Definition 4.

Problem 5 (L. Molnár)**.** *Is the geometric mean $\#$ the only Molnár mean?*

In [24, p. 161], L. Molnár says the following, in regard to the question above, which motivated our paper: "However, we still do not know if the answer to the question is positive or negative. If it were affirmative, then we would get an interesting common characterization of the three fundamental operator means, the arithmetic, harmonic and geometric means." We are able to answer his question (in the negative) by proving the following theorem:

Theorem 6. *There are infinitely many Molnár means.*

The main goal of our paper is to prove this and, furthermore, to completely characterize the class of Molnár means in terms of their representing functions, which we do with Theorems 16 and 17. These theorems are stated and proved in Section 3; our approach is illustrated graphically in Fig. 1. Finally, we delve deeper into the order structure of the the Molnár class (see Theorem 19) and give several infinite families of fully explicit nontrivial Molnár means in Section 4. We conclude the paper by a modification of Molnár's original characterization so that the geometric mean is the only one satisfying the requisite set of properties (see Theorem 20).

2 Characterization of the Molnár class of means

In this section we completely characterize the Molnár class of means in terms of their representing functions. The next lemma is an important first step in this regard (whose proof is immediate from our discussion above) and motivates the definition that follows. Also, as mentioned in the introduction, we will abuse notation and not distinguish between elements of OM_+ and positive operator monotone functions, unless necessary.

Lemma 7. *A connection σ with representing function f (i.e., $\sigma = \sigma_f$) is a Molnár mean if and only if all of the following statements hold:*

(i) $f \in OM_+$;

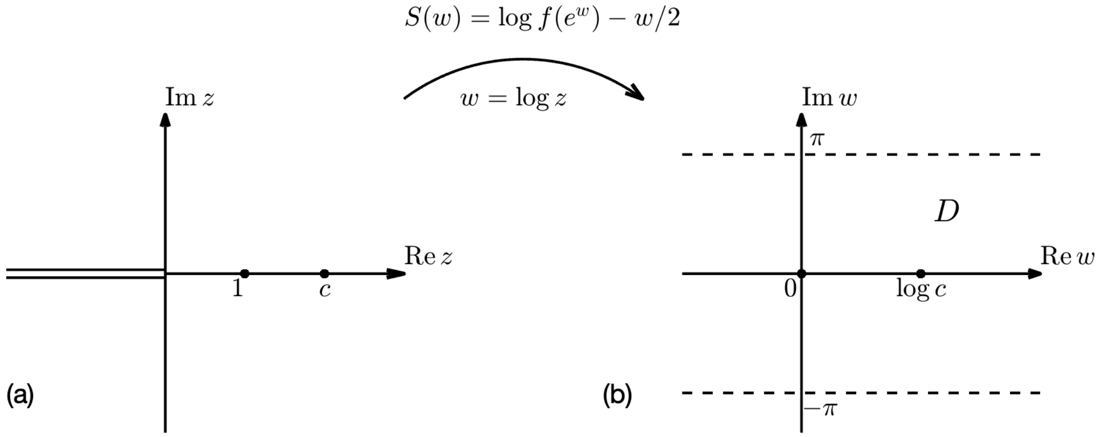

Figure 1: A Molnár mean $\sigma = \sigma_f$ corresponds to a representing function $f \in \mathscr{M}_c$ for some $c \in (1, \infty)$. As represented in the transition from (a) to (b) in the figure, by using the invertible transform $w = \log z$ (with inverse $z = e^w$) from $\mathbb{C} \setminus (-\infty, 0]$ onto the strip $D = \{w \in \mathbb{C} : |\operatorname{Im} w| < \pi\} = \mathbb{R} \times (-\pi, \pi)$, the analytic functions $f \in \mathscr{M}_c$ are mapped bijectively onto the analytic functions $S \in \mathscr{W}_p$ via $W_p(f) = \log f(e^w) - w/2$, where $p = 2\log c$. Then using the exponential representation of $OM_+ \setminus \{0\}$ functions from [16] we obtain the explicit representation of the class $\mathscr{W}_p$ and, perforce, of $\mathscr{M}_c$.

(ii) $f(1) = 1$*;*

(iii) $xf(1/x) = f(x)$, *for* $x > 0$*;*

(iv) there exists a positive scalar $c \neq 1$ *such that* $f(c^2 x) = cf(x)$*, for* $x > 0$.

Definition 8 (Molnár class of functions)**.** *A function f having properties (i)-(iv) in Lemma 7 will be called a Molnár function. For a positive scalar $c \neq 1$, we say f is a Molnár function of type c if it is a Molnár function and $f(c^2x) = cf(x)$, for $x > 0$. The set of all Molnár functions of type c will be denoted by $\mathscr{M}_c$ and the Molnár class of functions is $\mathscr{M} \equiv \cup_{c \in (0,\infty)\setminus\{1\}} \mathscr{M}_c$.*

As a consequence of the following lemma, to characterize the Molnár means class $\mathscr{M}$, we need to consider only $c \in (1, \infty)$.

Lemma 9. *For any positive scalar $c \neq 1$,*

$$\mathscr{M}_c = \mathscr{M}_{1/c}.$$

In particular,

$$\mathscr{M} = \cup_{c \in (1,\infty)} \mathscr{M}_c = \cup_{c \in (0,1)} \mathscr{M}_c.$$

Proof. Let c be a positive scalar $c \neq 1$. If $f(c^2x) = cf(x)$ for all $x > 0$ then $x > 0$ implies $c^{-2}x > 0$ and hence $f(x) = f(c^2(c^{-2}x)) = cf(c^{-2}x)$ so that $f((c^{-1})^2 x) = c^{-1}f(x)$ for all $x > 0$. $\square$

Next, we denote the principal branches of the logarithm and the square root by $\log(\cdot)$ and $\sqrt{\cdot}$, respectively, i.e.,

$$\log(z) = \log|z| + i\,\mathrm{Arg}(z),\ \mathrm{Arg}(z) \in (-\pi,\pi),\ z \in \mathbb{C} \setminus (-\infty,0],$$
$$\sqrt{z} = e^{\frac{1}{2}\log(z)},\ z \in \mathbb{C} \setminus (-\infty,0].$$

Now have some preliminary results.

Lemma 10. *Let $f \in OM_+ \setminus \{0\}$. Then*

$$\overline{f(\overline{z})} = f(z) \text{ and } f(z) \neq 0, \text{ if } z \in \mathbb{C} \setminus (-\infty,0].$$

In addition,

$$0 \leq \mathrm{Arg}\, f(z) \leq \mathrm{Arg}\, z < \pi,\ \text{if } \mathrm{Im}\, z > 0. \tag{2.1}$$

Proof. Let $f \in OM_+ \setminus \{0\}$. First, as $\overline{f(\overline{x})} = f(x)$ for all $x > 0$, it follows by analyticity that $\overline{f(\overline{z})} = f(z)$ for all $z \in \mathbb{C} \setminus (-\infty,0]$. Next, it follows from the integral representation (1.6) of f that $f(x) > 0$ for all $x > 0$. It also follows from the open mapping principle that, since $\Im(f(z)) \geq 0$ for all $z \in \mathbb{C}^+$, then either $f(z)$ is a positive constant, or $\Im(f(z)) > 0$, for all $z \in \mathbb{C}^+$. Inequality (2.1) is proved by invoking a dual connection (cf. [17, Corollary 4.3] and [14, p. 194]) $f_\perp \in OM_+ \setminus \{0\}$, defined by:

$$f_\perp(z) = \frac{z}{f(z)},\ \text{for } z \in \mathbb{C} \setminus (-\infty,0].$$

As $f, f_\perp \in OM_+ \setminus \{0\}$, then the compositions $\log \circ f$ and $\log \circ f_\perp$ are analytic functions on $\mathbb{C} \setminus (-\infty,0]$ satisfying $(\log \circ f_\perp)(x) = \log(x) - \log f(x)$ for all $x > 0$. By analyticity this implies $\log \circ f_\perp = \log - \log \circ f$ on $\mathbb{C} \setminus (-\infty,0]$ so that

$$0 \leq \mathrm{Im}[\log f_\perp(z)] = \mathrm{Im}(\log z) - \mathrm{Im}[\log f(z)] = \mathrm{Arg}\, z - \mathrm{Arg}\, f(z)$$

for every $z \in \mathbb{C}^+$, proving inequalities (2.1). □

Lemma 11. *If $f \in OM_+ \setminus \{0\}$ then the function $S : D \to \mathbb{C}$ defined by*

$$S(w) = \log f(e^w) - \frac{1}{2}w,\ \text{for } w \in D, \tag{2.2}$$
$$D = \{w \in \mathbb{C} : -\pi < \mathrm{Im}\, w < \pi\}, \tag{2.3}$$

is analytic on D and satisfies

$$|\mathrm{Im}\, S(w)| \leq \frac{1}{2}\mathrm{Im}\, w,\ \text{if } 0 \leq \mathrm{Im}\, w < \pi, \tag{2.4}$$
$$f(z) = \sqrt{z}e^{S(\log z)},\ \text{for } z \in \mathbb{C} \setminus (-\infty,0]. \tag{2.5}$$

Conversely, if $S : D \to \mathbb{C}$ is an analytic function satisfying (2.4) then the function $f : \mathbb{C} \setminus (-\infty,0] \to \mathbb{C}$ defined by (2.5) is in $OM_+ \setminus \{0\}$ and S is given in terms of f by (2.2).

Proof. (⇒): Let $f \in OM_+ \setminus \{0\}$. Then by Lemma 10, $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is an analytic function with range $f(\mathbb{C} \setminus (-\infty, 0]) \subseteq \mathbb{C} \setminus (-\infty, 0]$ so that the function $z \mapsto \log f(z)$ is analytic on the domain $\mathbb{C} \setminus (-\infty, 0]$. Next, the function $w \mapsto e^w$ is analytic on the domain D with range $e^D = \mathbb{C} \setminus (-\infty, 0]$. It follows that the composition of functions $w \mapsto \log f(e^w)$ is well-defined and analytic on D. From this it follows immediately that the function $S : D \to \mathbb{C}$ defined by (2.2), (2.3) is analytic on D and satisfies (2.5). Also, if $\operatorname{Im} w = 0$ then $w \in D$ and since $f(x) \geq 0$ for $x > 0$, then by (2.2) we have $\operatorname{Im} S(w) = \operatorname{Arg} f(e^{\operatorname{Re} w}) = 0$. To complete the proof of (2.4), we note that by the hypotheses on f it follows from Lemma 10 that $0 \leq \operatorname{Arg} f(z) \leq \operatorname{Arg} z < \pi$ if $\operatorname{Im} z > 0$ and since

$$\operatorname{Arg} e^w = \operatorname{Im} w, \ \operatorname{Im} e^w = e^{\operatorname{Re} w} \sin(\operatorname{Im} w) > 0, \text{ if } 0 < \operatorname{Im} w < \pi,$$

it follows that

$$-\frac{1}{2} \operatorname{Im} w \leq \operatorname{Im} S(w) = \operatorname{Arg} f(e^w) - \frac{1}{2} \operatorname{Im} w \leq \frac{1}{2} \operatorname{Im} w, \text{ if } 0 < \operatorname{Im} w < \pi,$$

which proves (2.4).

(⇐): Conversely, suppose $S : D \to \mathbb{C}$ is an analytic function [where D is defined by (2.3)] satisfying (2.4). Let $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ be the function defined by (2.5). As $\log z$ and $\sqrt{z}$ are analytic on the domain $\mathbb{C} \setminus (-\infty, 0]$ with $\log z \in D$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ then it follows that $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is analytic and cannot be the zero function since $f(z) = \sqrt{z} e^{S(\log z)} \neq 0$ for all z in the domain of f. Next, (2.4) implies that $S(\log x) \in \mathbb{R}$ for $x > 0$ so that $f(x) = \sqrt{x} e^{S(\log x)} > 0$ for $x > 0$. Also, (2.4) implies that for $\operatorname{Im} z > 0$ we have

$$0 \leq \operatorname{Im} \left[S(\log z) + \frac{1}{2} \log z \right] \leq \operatorname{Im} \log z = \operatorname{Arg} z < \pi \tag{2.6}$$

and hence

$$f(z) = \sqrt{z} e^{S(\log z)} = e^{S(\log z) + \frac{1}{2} \log z},$$
$$\operatorname{Im} f(z) = e^{\operatorname{Re}[S(\log z) + \frac{1}{2} \log z]} \sin \left\{ \operatorname{Im} \left[S(\log z) + \frac{1}{2} \log z \right] \right\} \geq 0.$$

Now to complete the proof, it remains only to prove that the function $S : D \to \mathbb{C}$ is given in terms of f by the formula (2.2). It follows by (2.5) that there exists a constant $m \in \mathbb{Z}$ such that

$$\log f(z) = S(\log z) + \frac{1}{2} \log z + i2\pi m, \text{ for } z \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.7}$$

This implies that

$$2\pi m + \operatorname{Im} \left[S(\log z) + \frac{1}{2} \log z \right] = \operatorname{Im} \log f(z) = \operatorname{Arg} f(z), \text{ for } z \in \mathbb{C} \setminus (-\infty, 0].$$

As we know that if $\operatorname{Im} z > 0$ then (2.6) holds and $\operatorname{Arg} f(z) \in [0, \pi)$, this implies $m = 0$. Hence, from (2.7) it follows that

$$\log f(e^w) = S(\log e^w) + \frac{1}{2} \log e^w = S(w) + \frac{1}{2} w, \text{ for } w \in D,$$

which proves equality (2.2). This completes the proof. □

Lemma 12. *Let $f \in OM_+ \setminus \{0\}$ and denote by $S : D \to \mathbb{C}$ the corresponding function defined by (2.2). Then the following statements are true:*

(i) $f(1) = 1$ *if and only if* $S(0) = 0$.

(ii) $f(x) = xf(1/x)$, *for* $x > 0$ *if and only if* $S(-w) = S(w)$, *for* $w \in D$.

(iii) *If* $c \in (1, \infty)$ *then* $f(c^2 x) = cf(x)$, *for* $x > 0$ *if and only if* $S(w + 2\log c) = S(w)$, *for* $w \in D$ [*i.e.,* S *is periodic with a period* $2\log c$], *in which case*

$$\lim_{x\to 0^+} f(x) = 0.$$

Proof. (i): If $f(1) = 1$ then $S(0) = \log f(e^0) - \frac{1}{2}(0) = \log 1 = 0$. Conversely, if $S(0) = 0$ then $f(1) = \sqrt{1}e^{S(\log 1)} = e^{S(0)} = 1$.

(ii): Suppose $f(x) = xf(1/x)$ for all $x > 0$. If $w \in \mathbb{R}$ then

$$\begin{aligned} S(-w) &= \log f(e^{-w}) - \frac{1}{2}(-w) = \log[e^{-w} f(e^w)] + \frac{1}{2}w \\ &= \log[f(e^w)] + \log[e^{-w}] + \frac{1}{2}w = \log[f(e^w)] - \frac{1}{2}w = S(w). \end{aligned}$$

It follows that $S(w) = S(-w)$ for all $w \in D$, since functions $S(w)$ and $S(-w)$ are analytic in D and agree on $\mathbb{R} \subset D$. Conversely, suppose that $S(w) = S(-w)$ for all $w \in D$. Then, for every $x > 0$,

$$xf(1/x) = x\sqrt{x^{-1}}e^{S(\log(x^{-1}))} = \sqrt{x}e^{S(-\log x)} = \sqrt{x}e^{S(\log x)} = f(x).$$

(iii): Let $c \in (1, \infty)$. Suppose $f(c^2x) = cf(x)$ for all $x > 0$. If $w \in \mathbb{R}$ then

$$\begin{aligned} S(w + 2\log c) &= \log f(e^{w+2\log c}) - \frac{1}{2}(w + 2\log c) \\ &= \log f(c^2 e^w) - \log c - \frac{1}{2}w = \log[cf(e^w)] - \log c - \frac{1}{2}w \\ &= \log c + \log f(e^w) - \log c - \frac{1}{2}w = S(w). \end{aligned}$$

It follows that $S(w+2\log c) = S(w)$ for all $w \in D$, since functions $S(w)$ and $S(w+2\log c)$ are analytic in D and agree on $\mathbb{R} \subset D$. Conversely, suppose $S(w + 2\log c) = S(w)$ for all $w \in D$. Then, for every $x > 0$,

$$f(c^2x) = \sqrt{c^2x}e^{S(\log(c^2x))} = c\sqrt{x}e^{S(\log x+2\log c)} = c\sqrt{x}e^{S(\log x)} = cf(x).$$

Thus, in particular, if this is the case, then the restriction $S|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function on $\mathbb{R}$, implying it is bounded, so that $x \mapsto e^{S(\log x)}$ is a bounded function on $(0, \infty)$ and therefore,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x}e^{S(\log x)} = 0.$$

(For alternative proof of the fact $\lim_{x\to 0^+} f(x) = 0$, cf. [24, pp. 160-161].) This completes the proof. □

These results motivate the following definition and proposition.

Definition 13 ($\mathscr{W}$-functions)**.** *Let $D = \{w \in \mathbb{C} : -\pi < \operatorname{Im} w < \pi\}$. An analytic function $S : D \to \mathbb{C}$ having the properties:*

(i) $|\operatorname{Im} S(w)| \leq \frac{1}{2} \operatorname{Im} w,$ *if* $0 \leq \operatorname{Im} w < \pi$,

(ii) $S(0) = 0$,

(iii) $S(-w) = S(w)$ *for all* $w \in D$,

(iv) there exists a scalar $p > 0$ *such that* $S(w + p) = S(w)$ *for all* $w \in D$,

will be called a $\mathscr{W}$-function with period p. The set of all $\mathscr{W}$-functions with period p will be denoted by $\mathscr{W}_p$ and the class of all $\mathscr{W}$-functions is $\mathscr{W} = \cup_{p\in(0,\infty)}\mathscr{W}_p$.

Proposition 14. *For each $f \in \mathscr{M}$, denote by S_f the function S defined by* (2.2) *and* (2.3)*. Then the map*

$$W : \mathscr{M} \to \mathscr{W},$$
$$W(f) = S_f,\ f \in \mathscr{M}$$

is a bijection from $\mathscr{M}$ onto $\mathscr{W}$ such that

$$f \in \mathscr{M}_c \Leftrightarrow S_f \in \mathscr{W}_p,$$

where $c \in (1, \infty), p \in (0, \infty)$ are related by the formula:

$$p = 2 \log c.$$

In particular, the restriction map W_p defined by

$$W_p = W|_{\mathscr{M}_c} : \mathscr{M}_c \to \mathscr{W}_p,$$
$$W_p(f) = W(f) = S_f,\ f \in \mathscr{M}_c$$

is a bijection from $\mathscr{M}_c$ onto $\mathscr{W}_p$.

Proof. The proof of this proposition follows immediately from Definitions 8 and 13 and Lemmas 7, 11, and 12. □

3 Explicit characterization of $\mathscr{W}_p$

The goal of this section is to characterize the class of all $\mathscr{W}_p$-functions explicitly. Before we do this, we will need the following well-known collection of results (see, for instance, [7, 8, 13, 10, 30]) on the relationship between Herglotz functions and their boundary values.

Theorem 15 (Herglotz integral representation)**.** *If $a \in \mathbb{R}, b \geq 0$, and μ is a positive Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1+\lambda^2)^{-1}\mu(\lambda) < \infty$ then*

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\mu(\lambda), \quad z \in \mathbb{C}^+.$$

is a Herglotz function. The map $(a, b, \mu) \mapsto h$ defines a bijection between the class of all such triples and the set of all Herglotz functions. In particular, the triple (a, b, μ) can be recovered from the Herglotz function h by the formulas

$$a = \operatorname{Re} h(i), \quad b = \lim_{\eta \uparrow \infty} \frac{h(i\eta)}{i\eta},$$
$$\frac{1}{2}\mu(\{\lambda_1\}) + \frac{1}{2}\mu(\{\lambda_2\}) + \mu((\lambda_1, \lambda_2)) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} h(\lambda + i\varepsilon) d\lambda, \quad (\lambda_1, \lambda_2) \subseteq \mathbb{R}.$$

Moreover, $h(z)$ has finite normal limits $\lim_{\varepsilon \downarrow 0} h(\lambda + i\varepsilon)$ for Lebesgue a.e. $\lambda \in \mathbb{R}$ and the absolutely continuous (ac) part μ_{ac} of the measure μ with respect to the Lebesgue measure on $\mathbb{R}$ has density function (i.e., Radon-Nikodym derivative)

$$\frac{d\mu_{ac}(\lambda)}{d\lambda} = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} h(\lambda + i\varepsilon), \ \textit{for Lebesgue a.e. } \lambda \in \mathbb{R}.$$

Furthermore, this latter statement is also true if we replace the normal limit to λ by the limit to λ in any (non-tangential) circular sector in the upper-half plane.

Our next result is an integral representation that shows that the class $\mathscr{W}_p$ can be parametrized by all measurable, real-valued, odd, periodic functions bounded by 1/2 in absolute value.

Theorem 16. *If $\Psi \in L^\infty(\mathbb{R})$ is a real-valued, odd, p-periodic function with $\|\Psi\|_\infty \leq 1/2$ then*

$$S(w) = \frac{1}{2} \int_{\mathbb{R}} \frac{\Psi(\lambda) \sinh\left(\frac{w}{2}\right) d\lambda}{\cosh\left(\frac{\lambda}{2}\right) \cosh\left(\frac{w-\lambda}{2}\right)}, \quad w \in D \tag{3.1}$$

is a function belonging to $\mathscr{W}_p$. The map $\Psi \mapsto S$ defines a bijection between the class of all such functions in $L^\infty(\mathbb{R})$ (with equality in the sense of the $|| \cdot ||_\infty$ norm) and the set $\mathscr{W}_p$. In particular, if $S \in \mathscr{W}_p$ then Ψ can be uniquely recovered by the formula:

$$\Psi(\lambda) = \frac{1}{\pi} \lim_{\mu \to \pi^-} \operatorname{Im} S(\lambda + i\mu), \ \textit{for Lebesgue a.e. } \lambda \in \mathbb{R}. \tag{3.2}$$

Proof. We start with the exponential representation of a function $f \in OM_+ \setminus \{0\}$ from [16, Appendix, Sec. 5] (see also [7, 13]): There exists scalar C and function $\phi \in L^\infty((0, \infty))$ such that for every $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$f(z) = Ce^{g(z)}, \quad C > 0, \quad g(z) = \int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{t+z} \right) \phi(t) dt, \quad 0 \leq \phi(t) \leq 1. \tag{3.3}$$

We observe that if $z \notin \mathbb{R}$ then

$$|\operatorname{Im} g(z)| = |\operatorname{Im} z| \int_0^\infty \frac{\phi(t)dt}{|t+z|^2} < \int_{\mathbb{R}} \frac{|\operatorname{Im} z|dt}{|t+z|^2} = \pi.$$

Thus, for all $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$S(\log z) = \log f(z) - \frac{1}{2}\log z = \log C - \frac{1}{2}\log z + g(z).$$

Observing that

$$\int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{t+z}\right) dt = \log z,$$

we obtain

$$S(\log z) = \log C + \int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{t+z}\right) \psi(t)dt, \quad ||\psi||_\infty \leq \frac{1}{2}, \tag{3.4}$$

where $\psi(t) = \phi(t) - 1/2$. For S to be in $\mathscr{W}_p$ we also need $S(0) = 0$, $S(-w) = S(w)$ and $S(w+p) = S(w)$. Clearly, $S(0) = 0$ is equivalent to

$$\log C = -\int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{t+1}\right) \psi(t)dt.$$

Hence,

$$S(\log z) = \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{t+z}\right) \psi(t)dt = \int_0^\infty \frac{(z-1)\psi(t)}{(t+1)(t+z)}dt,$$

and $S(-w) = S(w)$ is equivalent to

$$\int_0^\infty \frac{(z-1)\psi(t)}{(t+1)(t+z)}dt = \int_0^\infty \frac{(1-z)\psi(t)}{(t+1)(zt+1)}dt. \tag{3.5}$$

Making a change of variables $s = 1/t$ in the integral on the right-hand side in (3.5), and denoting $\widetilde{\psi}(s) = \psi(1/s)$, equation (3.5) becomes

$$\int_0^\infty \frac{\psi(t)}{(t+1)(t+z)}dt = -\int_0^\infty \frac{\widetilde{\psi}(s)}{(s+1)(z+s)}ds,$$

which implies $\psi(t) = -\widetilde{\psi}(t) = -\psi(1/t)$ for a.e. $t \in (0, \infty)$ (by Theorem 15). Hence, writing

$$S(\log z) = \int_0^1 \frac{(z-1)\psi(t)}{(t+1)(t+z)}dt + \int_1^\infty \frac{(z-1)\psi(t)}{(t+1)(t+z)}dt,$$

and changing the variable of integration $s = 1/t$ in the second term, we obtain

$$S(\log z) = \int_0^1 \frac{(z-1)\psi(t)}{(t+1)(t+z)}dt - \int_0^1 \frac{(z-1)\psi(s)}{(s+1)(1+zs)}ds.$$

Equivalently, we can write, using partial fraction decomposition,

$$S(\log z) = \int_0^1 \left(\frac{2}{t+1} - \frac{1}{t+z} - \frac{z}{1+tz} \right) \psi(t)dt. \tag{3.6}$$

Finally, $S(w+p) = S(w)$ is equivalent to

$$\int_0^1 \left(\frac{1}{t+z} + \frac{z}{1+tz} \right) \psi(t)dt = \int_0^1 \left(\frac{1}{t+c^2z} + \frac{c^2z}{1+tc^2z} \right) \psi(t)dt,$$

where $p = 2\log c$. Putting everything on one side, we obtain

$$\int_0^1 \left(\frac{1}{t+c^2z} + \frac{c^2z}{1+tc^2z} - \frac{1}{t+z} - \frac{z}{1+tz} \right) \psi(t)dt = 0, \quad \forall z \in \mathbb{C} \setminus (-\infty, 0]. \tag{3.7}$$

To understand (3.7) we want to rewrite it, so that each term above is a Cauchy-type integral. We accomplish this by changing variables $t = c^2 s$ in the first term, and $t = c^{-2}s$, in the second. Then, upon switching z to $-z$, we obtain the following identity on $\mathbb{C} \setminus [0, \infty)$:

$$F(z) := F_1(z) + F_2(z) = 0,$$

where

$$F_1(z) = \int_0^1 \frac{\psi(c^2s)\chi_{[0,c^{-2}]}(s) - \psi(s)}{s-z} ds, \ F_2(z) = \int_0^{c^2} \frac{\psi(sc^{-2}) - \psi(s)\chi_{[0,1]}(s)}{s - z^{-1}} ds.$$

It now follows from this (and by Theorem 15) that for a.e. $x \in (0, \infty)$:

$$0 = \frac{1}{\pi} \lim_{y \downarrow 0} \operatorname{Im} F(x+iy) = \begin{cases} \psi(xc^2) - \psi(x), & x \in (0, c^{-2}), \\ -\psi(x) - \psi((c^2x)^{-1}), & x \in (c^{-2}, 1), \\ \psi(x^{-1}) - \psi((c^2x)^{-1}), & x \in (1, \infty). \end{cases} \tag{3.8}$$

Now define $\Psi(\lambda) = \psi(e^\lambda)$ for $\lambda \in \mathbb{R}$. Then, writing $p = 2\log c$, we obtain from (3.8) that the following equations hold for a.e. $\lambda \in \mathbb{R}$:

$$\begin{cases} \Psi(\lambda+p) = \Psi(\lambda), & \lambda < -p, \\ \Psi(\lambda) = -\Psi(-\lambda-p), & -p < \lambda < 0, \\ \Psi(-\lambda) = \Psi(-\lambda-p), & \lambda > 0. \end{cases} \tag{3.9}$$

Recalling that $\psi(x) = -\psi(1/x)$, for a.e. $x > 1$, we obtain $\Psi(\lambda) = -\Psi(-\lambda)$ for a.e. $\lambda > 0$. It follows that $\Psi \in L^\infty(\mathbb{R})$ is a real-valued, odd, p-periodic function with $\|\Psi\|_\infty \le 1/2$. Finally, by substituting in $z = e^w, w \in D$ into (3.5) and changing variables $t = e^\lambda$, we arrive at the representation (3.1) for $S(w)$.

We now claim that any such function $\Psi(\lambda)$ implies that $F(z) = 0$ for all $z \in \mathbb{C} \setminus [0, \infty)$. Indeed, changing variables under the integral $s = e^\lambda$, we obtain

$$F(z) = \int_{-\infty}^0 \frac{\Psi(\lambda+p)\chi_{(-\infty,-p)}(\lambda) - \Psi(\lambda)}{1 - e^{-\lambda}z} d\lambda + \int_{-\infty}^p \frac{\Psi(\lambda-p) - \Psi(\lambda)\chi_{(-\infty,0)}(\lambda)}{1 - e^{-\lambda}z^{-1}} d\lambda.$$

Using the properties of the function $\Psi(\lambda)$ we obtain

$$F(z) = -\int_{-p}^{0} \frac{\Psi(\lambda)}{1-e^{-\lambda}z}d\lambda + \int_{0}^{p} \frac{\Psi(\lambda)}{1-e^{-\lambda}z^{-1}}d\lambda$$

Changing variables $\lambda = -\lambda'$ in the first integral and using the fact that $\Psi(\lambda)$ is odd, we obtain

$$F(z) = \int_0^p \Psi(\lambda)\left(\frac{1}{1-e^{\lambda}z} + \frac{1}{1-(e^{\lambda}z)^{-1}}\right)d\lambda = \int_0^p \Psi(\lambda)d\lambda.$$

Since $\Psi(\lambda)$ is odd and p-periodic we have

$$\int_0^p \Psi(\lambda)d\lambda = \int_{-p/2}^{p/2} \Psi(\lambda)d\lambda = 0.$$

This proves the claim. We can also conclude from this that for any $\Psi \in L^{\infty}(\mathbb{R})$ which is a real-valued, odd, $p = 2\log c$-periodic function such that $||\Psi|||_{\infty} \leq 1/2$, the function

$$f(z) = \sqrt{z}\exp\left\{(z-1)\int_{\mathbb{R}} \frac{\Psi(\lambda)}{(1+e^{\lambda})(1+e^{-\lambda}z)}d\lambda\right\}, \quad z \in \mathbb{C}\setminus(-\infty, 0] \tag{3.10}$$

is in the class $\mathscr{M}_c$ and the function $S(w)$ given by (3.1) belongs to $\mathscr{W}_p$.

Finally, given $S(w)$ in $\mathscr{W}_p$ we can compute the corresponding spectral function $\Psi(\lambda) = \psi(e^{\lambda})$ using the representation (3.4):

$$-S(\log(-z)) = -\log C + \int_0^{\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\psi(t)dt, \quad z \in \mathbb{C}\setminus[0,\infty).$$

Then it follows from this (by Theorem 15) that

$$\psi(x) = \frac{-1}{\pi}\lim_{\eta\to0^+}\operatorname{Im} S(\log(-xe^{i\eta})) = \frac{-1}{\pi}\lim_{\eta\to0^+}\operatorname{Im} S(\log x + i(\eta-\pi)) = \frac{1}{\pi}\lim_{\eta\to0^+}\operatorname{Im} S(\log x + i(\pi-\eta))$$

for a.e. $x \in (0,\infty)$, where we also used the identity $\overline{S(w)} = S(\overline{w})$ for all $w \in D$. Hence,

$$\psi(x) = \frac{1}{\pi}\lim_{\mu\to\pi^-}\operatorname{Im} S(i\mu + \log x) \tag{3.11}$$

for a.e. $x \in (0,\infty)$, from which formula (3.2) now follows. □

Theorem 16 shows that all classes $\mathscr{M}_c$, $c > 1$ have infinitely many members. However, the integral representation (3.1) does not permit us to exhibit functions in $\mathscr{M}_c$, or equivalently, in $\mathscr{W}_p$, explicitly. This is remedied by our next result.

Theorem 17. *Suppose*

$$\Psi(\lambda) = \sum_{n=1}^{\infty} B_n \sin(an\lambda) \tag{3.12}$$

is a Fourier series of a real, odd, $p = 2\pi/a$-periodic function $\Psi(\lambda)$, satisfying $|\Psi(\lambda)| \leq 1/2$. Then, the corresponding function $S \in \mathscr{W}_p$ in (3.1) is given by the Fourier series

$$S(w) = \pi\sum_{n=1}^{\infty} B_n \frac{1-\cos(awn)}{\sinh(a\pi n)} = 2\pi\sum_{n=1}^{\infty} B_n \frac{\sin^2\left(\frac{awn}{2}\right)}{\sinh(a\pi n)}. \tag{3.13}$$

Proof. We begin with the observation that the functions $S_n(w) = A_n(1 - \cos(awn))$, $n \in \mathbb{N}$, are entire, even, $p = 2\pi/a$-periodic, and satisfy $S_n(0) = 0$. Thus, $S_n \in \mathscr{W}_p$, if we can show that $S_n(w)$ has property (i) in Definition 13. We compute

$$\Im S_n(w) = A_n \sin(a\lambda n) \sinh(a\mu n), \quad w = \lambda + i\mu.$$

Since $\sinh(a\mu n)$ is convex on $\mu \in [0, \pi]$ we have

$$\sinh(a\mu n) \leq \frac{\sinh(a\pi n)}{\pi}\mu.$$

Since, $\sin(a\mu n)$ is an odd function of μ, we conclude that

$$|\Im S_n(w)| \leq |A_n| \frac{\sinh(a\pi n)}{\pi} |\Im w|.$$

Thus, choosing $A_n = \pi/(2\sinh(a\pi n))$, we obtain $S_n \in \mathscr{W}_p$. Formula (3.2) gives

$$\Psi_n(\lambda) = \frac{1}{2\sinh(a\pi n)} \lim_{\mu \to \pi^-} \sin(a\lambda n) \sinh(a\mu n) = \frac{1}{2} \sin(a\lambda n).$$

By linearity of the representation (3.1) we conclude that if Ψ is given by (3.12), then the corresponding $S(w)$ must be given by (3.13). By Theorem 16, if $|\Psi(\lambda)| \leq 1/2$, then $S \in \mathscr{W}_p$. $\square$

4 Conclusions

From our results, we are able to make several important conclusions. One is a corollary of Theorem 17 that provides a sequence of explicit non-geometric Molnár means, corresponding to $\Psi(\lambda) = \pm(1/2)\sin(\pi\lambda/\log c)$:

$$f_n(x) = \sqrt{x} \exp\left\{ \frac{\pi \sin^2\left(\frac{\pi n \log x}{2\log c}\right)}{\sinh\left(\frac{\pi^2 n}{\log c}\right)} \right\} \in \mathscr{M}_c, \quad n \in \mathbb{Z} \setminus \{0\}, \tag{4.1}$$

or recalling that the Molnár class $\mathscr{M}$ is the union of all $\mathscr{M}_c$, the family

$$f_\alpha(x) = \sqrt{x} \exp\left\{ \frac{\pi \sin^2(\alpha \log x)}{\sinh(2\pi\alpha)} \right\} \in \mathscr{M}, \quad \alpha \in \mathbb{R} \setminus \{0\}. \tag{4.2}$$

Second, we observe that any real, odd, p-periodic function $\Psi(\lambda)$, satisfying $|\Psi(\lambda)| \leq 1/2$ is uniquely determined by its restriction to the half-period interval $(0, p/2)$. Using the symmetries of $\Psi(\lambda)$, we can rewrite the exponential-integral representation (3.10) as

$$f(z) = \sqrt{z} \exp\left\{ \int_0^{p/2} \Psi(\lambda) E_p(\lambda; z) d\lambda \right\}, \quad \Psi \in B(0, 1/2) \subseteq L^\infty(0, p/2), \tag{4.3}$$

where for each fixed $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$E_p(\lambda; z) = \sum_{n\in\mathbb{Z}} \frac{(z-1)^2(e^{\lambda+pn}-1)}{(e^{\lambda+pn}+1)(z+e^{\lambda+pn})(z+e^{-(\lambda+pn)})}, \tag{4.4}$$

and where $B(0, 1/2) \subseteq L^\infty(0, p/2)$ denotes the closed ball centered at 0 and radius 1/2 in the Banach space $L^\infty(0, p/2)$. It is evident from formula (4.4) that $\lambda \mapsto E_p(\lambda; z)$ is an odd elliptic function with periods p and $2\pi i$ and three poles at πi, $\pm \log(-z)$ modulo periods in each period cell with residues 2, -1 and -1, respectively, provided these three points don't have congruent pairs modulo periods, i.e., $z \neq e^{pk/2}$, $k \in \mathbb{Z}$. The classical theory of elliptic functions gives the decomposition of $E_p(\lambda; x)$ in terms of the Weierstrass ζ-functions (see, e.g., [1, §14])

$$E_p(\lambda; z) = 2\zeta(\lambda + i\pi) - \zeta(\lambda + i\pi + \log z) - \zeta(\lambda + i\pi - \log z), \quad x > 0, \tag{4.5}$$

where $\zeta(u)$ is the Weierstrass ζ-function with periods p and $2\pi i$. This formula shows that $E_p(\lambda; z) = 0$, when $z = e^{pk}$, $k \in \mathbb{Z}$. If $z = e^{p/2}e^{kp} = ce^{kp}$, $k \in \mathbb{Z}$, then $E_p(\lambda; z)$ has only two poles at πi and $\pi i + p/2$ in the rectangle of periods. The residues at these poles are 2 and -2, respectively. Therefore,

$$E_p(\lambda; c) = \frac{2\sqrt{m}K'(m)}{\pi} \mathrm{sn}\left(\frac{K'(m)\lambda}{\pi}, m\right), \quad e^{\frac{2\pi K(m)}{K'(m)}} = c, \tag{4.6}$$

where $\mathrm{sn}(u, m)$ is a Jacobi elliptic sine function, $K(m)$ is a complete elliptic integral of the first kind, and $K'(m) = K(1 - m)$. Formula (4.6) was obtained by matching the poles at iK' and $2K + iK'$ of the Jacobi elliptic sine to the poles of $E_p(\lambda; c)$ and rescaling the residues $\pm 1/\sqrt{m}$ of $\mathrm{sn}(u, m)$ at the poles to match the residues ± 2 of $E_p(\lambda; c)$. The parameter m defined by the second equation in (4.6) was chosen to match the ratio of the two periods of $\mathrm{sn}(u, m)$ and $E_p(\lambda; c)$.

Lemma 18. *When $z \neq e^{pk/2}$, $k \in \mathbb{Z}$, the elliptic function $\lambda \mapsto E_p(\lambda; z)$ has exactly three simple zeros at $\lambda = 0$, $p/2$ and $p/2 + \pi i$. Moreover, the Fourier coefficients of the sine series of $E_p(\lambda; z)$ regarded as an odd p-periodic function of λ are*

$$S_n(w) = \int_0^{p/2} E_p(\lambda; e^w) \sin\left(\frac{2\pi n\lambda}{p}\right) d\lambda = 2\pi \frac{\sin^2\left(\frac{\pi n w}{p}\right)}{\sinh\left(\frac{2\pi^2 n}{p}\right)}. \tag{4.7}$$

Proof. Since $E_p(\lambda; z)$ has exactly 3 poles, it must have exactly 3 zeros, counting multiplicity, in the period rectangle. Moreover, by the Liouville theorem the sum of all three zeros, counting multiplicity, must be equal to the sum of poles, i.e., to πi, modulo periods. If $E_p(\lambda; z)$ has a single zero λ_0 of multiplicity 3, then $\lambda_0 = 0$, since $E_p(\lambda; z)$ is an odd function. But then, the sum of all zeros would be $0 \neq \pi i$. If $E_p(\lambda; z)$ has a double-zero at 0 and a simple zero at $\lambda_1 \neq 0$, then, by the Liouville theorem, $\lambda_1 = \pi i$ modulo periods, which is impossible, since πi is a pole of E_p. If 0 is a simple zero and λ_1 is a double zero, then $-\lambda_1$ must also be a double zero (E_p is an odd function of λ). Therefore, $-\lambda_1$ must be congruent to λ_1, and we must have $2\lambda_1 = 0$ modulo periods. But then the sum of all zeros $0 + \lambda_1 + \lambda_1 = 2\lambda_1 = 0 \neq \pi i$ modulo periods.

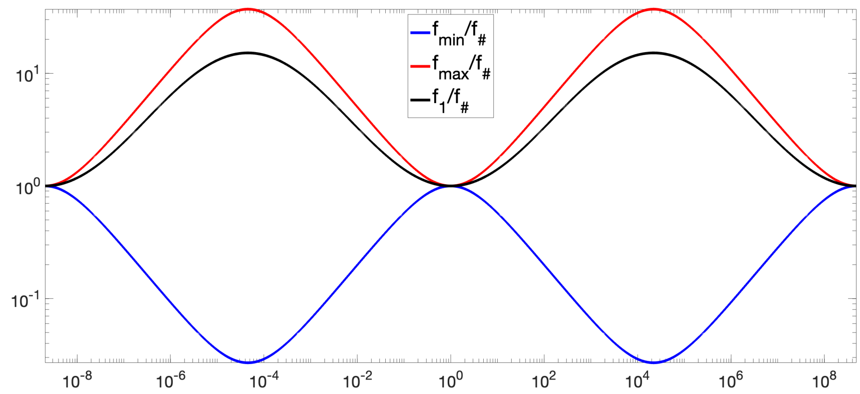

Figure 2: Graphs of the minimal and the maximal elements of $\mathscr{M}_c$ relative to $f_{\#}(x) = \sqrt{x}$ for $c = e^{10}$, corresponding to $p = 20$. All other functions in $\mathscr{M}_c$, such as f_1, given by (4.1) with $n = 1$, are sandwiched between $f_{\min}$ and $f_{\max}$.

Hence, $E_p(\lambda; z)$ must have three distinct simple zeros: 0, λ_1 and λ_2. Since E_p is odd, $-\lambda_1$ is also a zero of E_p. It is therefore must be congruent to either λ_1 or λ_2. In the latter case $\lambda_1 + \lambda_2 = 0$ modulo periods, and the sum of all zeros will be $0 + \lambda_1 + \lambda_2 = 0 \neq \pi i$ modulo periods. This contradiction shows that $-\lambda_1$ must be congruent to λ_1. Therefore, $-\lambda_2$ must be congruent to λ_2. Hence we have $2\lambda_1 = 2\lambda_2 = 0$ and $\lambda_1 + \lambda_2 = \pi i$, modulo periods. Equation $2\lambda = 0$ has only 3 nonzero solutions in the period rectangle: $\lambda = p/2$, $\lambda = i\pi$, and $\lambda = p/2 + i\pi$. This gives $\lambda_1 = p/2$ and $\lambda_2 = \pi i + p/2$, since πi is a pole.

Formula (4.7) is an immediate consequence of Theorem 17 that says that $\Psi(\lambda) = \frac{1}{2}\sin\left(\frac{2\pi n\lambda}{p}\right)$ corresponds to $S(w) = \pi\frac{\sin^2\left(\frac{\pi w n}{p}\right)}{\sinh(2\pi^2 n)}$, and (4.7) follows. □

Theorem 19 (Order structure of the Molnár class)**.** *The parametrization*

$$L^\infty(0, p/2) \supseteq B(0, 1/2) \ni \Psi \mapsto f \in \mathscr{M}_c \tag{4.8}$$

is order-preserving, and therefore any $f \in \mathscr{M}_c$ lies between the minimal and the maximal elements of $\mathscr{M}_c$ (see Fig. 2):

$$f_{\min}(x) = \sqrt{x}e^{-\frac{1}{2}\int_0^{p/2} E_p(\lambda;x)d\lambda} \leq f(x) \leq \sqrt{x}e^{\frac{1}{2}\int_0^{p/2} E_p(\lambda;x)d\lambda} = f_{\max}(x). \tag{4.9}$$

Moreover,

$$f_{\min}(x) = \frac{\sqrt{x}}{\sqrt{m}+1}\left(\mathrm{dn}\left(\frac{K'(m)\log x}{\pi}, m\right) + \sqrt{m}\,\mathrm{cn}\left(\frac{K'(m)\log x}{\pi}, m\right)\right) \in \mathscr{M}_c, \tag{4.10}$$

$$f_{\max}(x) = \frac{x}{f_{\min}(x)} = \frac{(\sqrt{m}+1)\sqrt{x}}{\mathrm{dn}\left(\frac{K'(m)\log x}{\pi}, m\right) + \sqrt{m}\,\mathrm{cn}\left(\frac{K'(m)\log x}{\pi}, m\right)} \in \mathscr{M}_c, \tag{4.11}$$

where $\mathrm{cn}(u, m)$ *and* $\mathrm{dn}(u, m)$ *are the Jacobi elliptic cosine and elliptic delta functions, and* $m \in (0, 1)$ *is the unique solution of* $4\pi K(m)/K'(m) = p = 2\log c$*. We also have*

$$\inf_{f\in\mathscr{M}} f(x) = f_!(x), \qquad \sup_{f\in\mathscr{M}} f(x) = f_\nabla(x). \tag{4.12}$$

where f_∇ *and* $f_!$ *are defined in (1.8) and (1.9), respectively.*

Proof. Lemma 18 implies that $E_p(\lambda; x)$, $x > 0$, $x \neq e^{kp}$, $k \in \mathbb{Z}$, is real and does not change sign on $\lambda \in (0, p/2)$. Moreover, $E_p(\lambda; x) > 0$, $x > 0$, $x \neq e^{kp}$, since $\sin(2\pi\lambda/p) > 0$ on $(0, p/2)$ and formula (4.7) shows that

$$\int_0^{p/2} E_p(\lambda; x) \sin\left(\frac{2\pi\lambda}{p}\right) d\lambda = 2\pi \frac{\sin^2\left(\frac{\pi \log x}{p}\right)}{\sinh\left(\frac{2\pi^2}{p}\right)} > 0,$$

when $x \neq e^{kp}$, $k \in \mathbb{Z}$. This implies that the parametrization (4.8) is order-preserving.

Using the expansion

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{2\sin\left(\frac{2\pi(2n+1)x}{p}\right)}{\pi(2n+1)}, \quad x \in (0, p/2),$$

corresponding to the odd p-periodic extension of $\Psi(\lambda) = 1/2$ on $(0, p/2)$, we obtain

$$S_*(w) := \frac{1}{2}\int_0^{p/2} E_p(\lambda; e^w) dx = \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)} \int_0^{p/2} E_p(\lambda; e^w) \sin\left(\frac{2\pi(2n+1)\lambda}{p}\right) d\lambda.$$

Using formula (4.7) we obtain

$$S_*(w) = \sum_{n=0}^{\infty} \frac{2\left(1 - \cos\left(\frac{2\pi(2n+1)w}{p}\right)\right)}{(2n+1)\sinh\left(\frac{2\pi^2(2n+1)}{p}\right)}.$$

We observe that

$$S_*'(w) = \frac{4\pi}{p} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{2\pi(2n+1)w}{p}\right)}{\sinh\left(\frac{2\pi^2(2n+1)}{p}\right)}.$$

It remains to notice that the Fourier sine series coefficients of $S_*'(w)$ are exactly the Fourier sine series coefficients of $(1/2)E_p(w; c)$. We conclude, using formula (4.6), that

$$S_*'(w) = \frac{1}{2}E_p(w; c) = \frac{\sqrt{m}K'(m)}{\pi} \mathrm{sn}\left(\frac{K'(m)w}{\pi}, m\right),$$

where $m \in (0, 1)$ is the unique solution of $4\pi K(m)/K'(m) = p$. Therefore,

$$S_*(w) = \int_0^w S_*'(v) dv = \sqrt{m} \int_0^{\frac{K'(m)w}{\pi}} \mathrm{sn}(u, m) du.$$

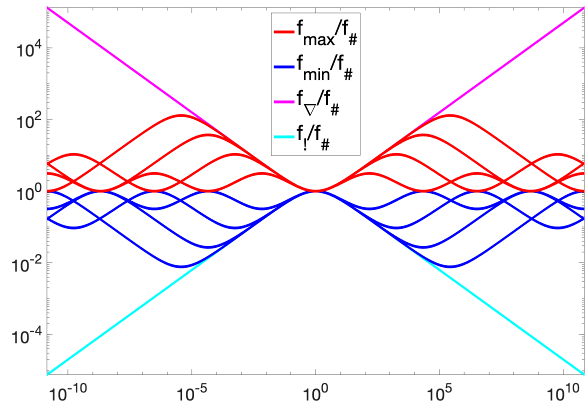

Figure 3: f_∇ and $f_!$ are the limits of $f_{\max}$ and $f_{\min}$, respectively, as $c \to \infty$. The plots of $f_{\max}/f_\#$ and $f_{\min}/f_\#$ for $p = 10, 15, 20, 25$ are shown.

Using the antiderivative formula given in [1, p. 215] we obtain

$$S_*(w) = \log\left(\frac{\sqrt{m}+1}{\mathrm{dn}\left(\frac{K'(m)w}{\pi}, m\right) + \sqrt{m}\,\mathrm{cn}\left(\frac{K'(m)w}{\pi}, m\right)}\right),$$

This proves formulas (4.10) and (4.11). In particular, $f_{\min}$ and $f_{\max}$, $m \in (0,1)$ are two other explicit infinite families of representing functions of non-geometric means in $\mathscr{M}$. Their plots for $p = 20$, relative to $f_\#$, given by (1.10) are shown in Fig. 2.

Since

$$\lim_{m\to 1^-} \mathrm{cn}(u,m) = \lim_{m\to 1^-} \mathrm{dn}(u,m) = \frac{1}{\cosh u},$$

we compute

$$\lim_{m\to 1^-} f_{\max}(x) = \frac{x+1}{2} = f_\nabla(x), \qquad \lim_{m\to 1^-} f_{\min}(x) = \frac{x}{f_\nabla(x)} = f_!(x). \tag{4.13}$$

According to [17, Theorem 4.5], the arithmetic mean is the maximum of all symmetric means, while the harmonic mean is the minimum. Since all Molnár means are symmetric, and in view of (4.9) and (4.13), we have

$$f_!(x) \le \inf_{f\in\mathscr{M}} f(x) = \inf_{c>1} \min_{f\in\mathscr{M}_c} f(x) = \inf_{m\in(0,1)} f_{\min}(x) \le f_!(x), \tag{4.14}$$

$$f_\nabla(x) \ge \sup_{f\in\mathscr{M}} f(x) = \sup_{c>1} \max_{f\in\mathscr{M}_c} f(x) = \sup_{m\in(0,1)} f_{\max}(x) \ge f_\nabla(x). \tag{4.15}$$

Therefore, all inequalities in (4.14) and (4.15) are equalities, and (4.12) is established. Fig. 3 illustrates this observation. □

Finally, we conclude the paper with a version of Lemma 7 that uniquely characterizes the geometric mean.

Theorem 20 (Characterization of the geometric mean)**.** *A connection σ with representing function f (i.e., $\sigma = \sigma_f$) is a geometric mean $\#$ if and only if all of the following statements hold:*

(i) $f \in OM_+$*;*

(ii) $f(1) = 1$*;*

(iii) $xf(1/x) = f(x)$*, for* $x > 0$*;*

(iv) there exist two logarithmically incommensurate positive scalars $c_1 \neq 1$*,* $c_2 \neq 1$ *(i.e.,* $\frac{\log c_1}{\log c_2} \notin \mathbb{Q}$*) such that* $f\left(c_1^2 x\right) = c_1 f\left(x\right)$ *and* $f\left(c_2^2 x\right) = c_2 f\left(x\right)$*, for* $x > 0$*.*

Proof. Our analysis shows that a connection σ satisfying all conditions of the theorem would correspond to an odd periodic function $\Psi(\lambda)$ that has two nonzero incommensurate periods $p_1 = 2\log c_1$ and $p_2 = 2\log c_2$. Therefore, $\Psi(\lambda) = 0$ identically and $f(z) = \sqrt{z}$, corresponding to the geometric mean $\#$. □

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