PHYS5702 Quantum Mechanics II Fall 2018 HW #6

Due at 5pm to the **Instructor** on Wednesday 12 Dec 2018

You are to work independently on this problem set. You are free to use whatever notes, books, computers, or other reference works you feel are useful. You are also free to consult the course instructor for help. You may *not*, however, consult other students in the class.

Please attach this page to your homework solutions along with your signature, below.

"I have complied with the requirement that I work independently on this problem set. I have not consulted with anyone other than the course instructor in preparing these solutions."

Signature: ______________________________

Print name: ______________________________

Please note that there are a total of six problems. All will be equally weighted.

HW #6 (Six Problems)

(1) The $n = 2$ state of hydrogen is eight-fold degenerate, accounting for both spin and orbital angular momentum. This degeneracy is broken by a perturbation

$$V = \frac{A}{\hbar^2}\mathbf{L}\cdot\mathbf{S} + \frac{B}{\hbar}(L_z + 2S_z)$$

where $\mathbf{L}$ and $\mathbf{S}$ are the orbital and spin angular momentum operators, A is a constant, and B is proportional (but not equal) to an applied magnetic field in the z-direction.

- **(a)** Write V in terms of $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$, J_z, and S_z, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$.
- **(b)** Find all nonzero matrix elements of V in the basis $|l, s = 1/2, j = l \pm 1/2, m\rangle$ for the eight $n = 2$ states. If you use formulas from the text, please cite them. *Hint:* Show that the 8×8 matrix decouples into four 2×2 matrices, two of which are diagonal.
- **(c)** Use degenerate perturbation theory to find the first order energy shifts Δ. **For all eight states**, plot Δ/A as a function of B/A. Explain why the resulting spectrum looks qualitatively different for $B/A \ll 1$ and $B/A \gg 1$.

(2) A spinless particle with charge e scatters from a spherically symmetric charge distribution $e\rho(r)$, where $\int_0^\infty 4\pi\rho(r)r^2 dr = 1$. Appropriately modify the Yukawa potential (6.3.8), and calculate the scattering amplitude in the Born approximation. Show that in the limit $q \to 0$ (with $\mu = 0$) the scattering amplitude separates into two terms, the second of which is proportional to $q^2\langle r^2\rangle$ where the mean square charge radius $\langle r^2\rangle \equiv \int_0^\infty 4\pi r^4\rho(r)dr$.

(3) An operator a obeys the anti-commutation relations $\{a, a\} = 0$ and $\{a, a^\dagger\} = 1$. Prove that the operator $N = a^\dagger a$ has eigenvalues 0 and 1.

(4) Three identical particles are in a one-dimensional harmonic oscillator potential well with classical angular frequency ω.

- **(a)** Write the complete time-independent Hamiltonian for this system, and express it in coordinate space as a differential equation whose solution is the three-body wave function $\Psi(x_1, x_2, x_3)$.
- **(b)** Assume the particles have zero spin. Use the single particle wave functions to construct the ground state wave function $\Psi_0(x_1, x_2, x_3)$, and show that it satisfies the differential equation in **(a)**, and find the ground state energy.
- **(c)** Assume the particles have spin-1/2. Repeat **(b)**, and also construct the ground state spin state from single particle spin eigenstates.

(5) A function $\Psi(\mathbf{x}, t)$ satisfies the Klein Gordon equation, namely

$$\left[\partial^\mu\partial_\mu + m^2\right]\Psi(\mathbf{x}, t) = 0$$

Use this to prove that function

$$\rho(\mathbf{x}, t) = C\left[\Psi^*\frac{\partial\Psi}{\partial t} - \left(\frac{\partial\Psi}{\partial t}\right)^*\Psi\right]$$

(where C is a constant) represents the density of a conserved quantity.

(6) Expand the energy eigenvalues given by (8.4.43) in powers of $Z\alpha$ and show that the result is equivalent to including the relativistic correction to the kinetic energy (5.3.10) and the spin-orbit interaction (5.3.31) to the nonrelativistic energies of the one-electron atom.