Notes on Complex Analysis in Physics

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These notes are meant to accompany a graduate level physics course, to provide a basic introduction to the necessary concepts in complex analysis. They are not complete, nor are any of the proofs considered rigorous. The immediate goal is to carry through enough of the work needed to explain the Cauchy Residue Theorem. Other material may be added later.

Complex Numbers and Complex Functions

A complex number z can be written as

$$z = x + iy \qquad \text{or} \qquad z = re^{i\phi} \ \ \text{with } r \geq 0$$

where $i = \sqrt{-1}$, and x, y, r, and ϕ are real numbers. Clearly, $x = r\cos\phi$ and $y = r\sin\phi$ leading to a description in terms of the "complex plane." The complex conjugate of z is

$$z^* = x - iy \qquad \text{or} \qquad z^* = re^{-i\phi}$$

The "modulus" of z is $|z| \equiv \sqrt{z^*z} = r = \sqrt{x^2+y^2}$ and ϕ is often called the "phase" of z.

A complex function $f(z)$ typically returns a complex number. Generically, we write

$$f(z) = u(x,y) + iv(x,y) \tag{1}$$

for purposes of proofs or illustrations. The behavior of the (real) functions $u(x,y)$ and $v(x,y)$ are critical for classifying complex functions, as seen when we consider taking derivatives.

Differentiation and Analyticity

We define the derivative $f'(z) = df/dz$ of a complex function $f(z)$ in the same was as we do for the derivatives of real functions. That is, for $z_0 \equiv x_0 + iy_0$,

$$f'(z_0) = \left.\frac{df}{dz}\right|_{z=z_0} = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

However, there is clearly an ambiguity, depending on whether we approach z_0 along the line $y = y_0$ or along $x = x_0$. (Of course, we could also say the ambiguity is along any line of constant $\phi = \phi_0$, but it is sufficient to consider just two orthogonal directions.) That is,

$$\begin{aligned} f'(z_0) &= \lim_{x\to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i \lim_{x\to x_0} \frac{v(x,y_0)-v(x_0,y_0)}{x-x_0} &&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \text{or} \qquad f'(z_0) &= \lim_{y\to y_0} \frac{u(x_0,y)-u(x_0,y_0)}{iy-iy_0} + i \lim_{y\to y_0} \frac{v(x_0,y)-v(x_0,y_0)}{iy-iy_0} &&= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned}$$

Therefore, in order to remove the ambiguity and have a consistent definition of the derivative,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

These are called the Cauchy-Riemann conditions. A function $f(z)$ which satisfies these rather restrictive conditions is called **analytic**. Indeed, analytic functions have very many applications in physics, and we will merely scratch the surface here.

For example, the function $f(z) = e^z = e^x(\cos y + i\sin y)$ is analytic. This is easy to prove. Putting $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$,

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

so the Cauchy-Riemann conditions (2) are satisfied.

It is simple to show that $f(z) = az$ is analytic, where a is a complex constant. It is also not hard to show that the product of two analytic functions is analytic, so any function of the form $f(z) = a_n z^n$, where n is a non-negative integer, is also analytic. Of course, any sum of analytic functions is analytic, so we see that any polynomial in z is analytic in the entire complex plane.

These examples beg the question: If a function $f(z)$ can be written explicitly in terms of z, is it analytic? The answer is "Yes." To see this, realize that instead of x and y, we could always write a complex function in terms of z and z^* using $x = (z+z^*)/2$ and $y = (z-z^*)/2i$. Now consider

$$\begin{aligned} \frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \end{aligned}$$

so long as the Cauchy-Riemann conditions (2) are satisfied. That is, if the expression for $f(z)$ contains only z (and not z^*) then the function is analytic.

There is some common terminology. A function $f(z)$ need not be analytic in the entire complex plane. (If it is, we called the function "entire.") If it is analytic at a point z_0 then we call that a "regular point." Otherwise, z_0 is called a "singular point." Much of our discussion of complex integration will focus on the notion of singular points.

Integration and Series Expansion

Similarly to differentiation, we approach integration of complex functions the same way as with real functions, but we need to be aware that there is now an arbitrariness of the "path" of integration. With $dz = dx + idy$ and using using (1), we have

$$\int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2} (u\ dx - v\ dy) + i\int_{z_1}^{z_2} (v\ dx + u\ dy) = \int_{z_1}^{z_2} \mathbf{A}\cdot d\mathbf{x} + i\int_{z_1}^{z_2} \mathbf{B}\cdot d\mathbf{x} \qquad (3)$$

where $\mathbf{A} = u\hat{\mathbf{x}} - v\hat{\mathbf{y}}$ and $\mathbf{B} = v\hat{\mathbf{x}} + u\hat{\mathbf{y}}$. So, we can now think of the two integrals on the right as real integrals of vector functions over curves in the xy plane. However, if we invoke Stokes' Theorem, these become integrals of the curls, and using (2), we find

$$\boldsymbol{\nabla}\times\mathbf{A} = \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \qquad \text{and} \qquad \boldsymbol{\nabla}\times\mathbf{B} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = 0 \qquad (4)$$

and each of the two integrals on the right in (3) are path-independent. Hence, the integral of an *analytic* complex function $f(z)$ is path-independent and can be unambiguously defined.

From here on, we assume all functions to be analytic unless explicitly noted otherwise. It is obvious from (3) that, when integrating around a closed path C,

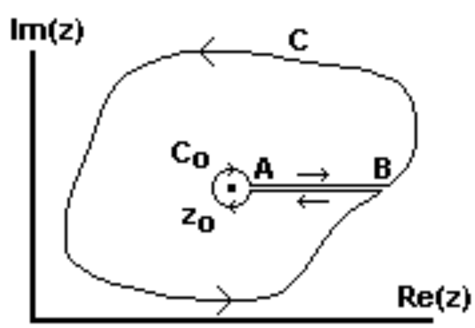
$$\oint_C f(z)dz = 0$$

which is known as the Cauchy-Goursat Theorem. We will be exploring circumstances where the integrand is explicitly singular at one or more points.

For the first example, we prove the Cauchy Integral Formula, namely

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz \qquad (5)$$

where C is a closed contour in the complex plane that contains the point z_0 and traversed in the counterclockwise direction. We can break up a contour C into something that looks like



where C_0 is a tiny *circular* contour around the singular point, but in the clockwise direction.

That is, we replace C with $\lim_{C_0 \to 0}(C + C_0)$. However, for $C_0 \neq 0$, the new contour C does not include the singular point, so by (4) we write (5) as

$$f(z_0) = -\frac{1}{2\pi i} \oint_{C_0} \frac{f(z)}{z - z_0} dz \tag{6}$$

The shrinking contour C_0 is parameterized as $z - z_0 = re^{i\phi}$ for $r \to 0$ and $\phi = 2\pi \to 0$, so

$$-\frac{1}{2\pi i} \oint_{C_0} \frac{f(z)}{z - z_0} dz = -\frac{1}{2\pi i} f(z_0) \int_{2\pi}^{0} \frac{1}{re^{i\phi}} ire^{i\phi} d\phi = -\frac{1}{2\pi i} f(z_0) i(-2\pi) = f(z_0)$$

proving the Cauchy Integral Formula (5). A trivial, but suggestive, rewriting of (5) gives

$$f(z) = \frac{1}{2\pi i} \oint_{C} \frac{f(\xi)}{\xi - z} d\xi \tag{7}$$

which leads to a convenient way to write the derivatives of a complex function, namely

$$f^{(n)}(z) = \frac{d^n f}{dz^n} = \frac{n!}{2\pi i} \oint_{C} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \tag{8}$$

Now consider the series expansion of an analytic function $f(z)$. We would naturally write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{9}$$

$$\text{where} \quad a_n \equiv \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_{C} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \tag{10}$$

Such a Taylor Series expansion works out as expected, but the curve C specifies regions in which the series converges.

This idea can be expanded to include $-\infty \leq n \leq \infty$, still using the right side of (10) to define a_n, and with modified regions of convergence. Such an expansion is called a Laurent Series. It clearly is not, in general, an analytic function because of poles that appear for $n < 0$. These, however, lead us to one of the most important theorems of complex analysis, so far as mathematical physics is concerned.

The Cauchy Residue Theorem

Let $g(z)$ have an isolated singularity at $z = z_0$. If the Laurent expansion can be written as

$$g(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{11}$$

then we say that $g(z)$ has a "simple pole" at $z = z_0$. Higher order poles are possible, but we're not going to consider them here.

Consider a contour C within the radius of convergence of $g(z)$. Separate the integral of $g(z)$ around this contour into two terms, one for each of the two terms on the right in (11). The second term is a polynomial in z; therefore it is analytic and the integral is zero. Recall that we reduced the contour to a small circle around the pole in order to prove the Cauchy Integral Formula. We can do the same thing here, and

$$\oint_C g(z)dz = b_1 \oint_C \frac{1}{z - z_0} = 2\pi i\ b_1 \tag{12}$$

We refer to b_1 as the "residue" of $g(z_0)$, sometimes written as $\text{Res}[g(z_0)]$. We have

$$\text{Res}[g(z_0)] = \lim_{z \to z_0} (z - z_0) g(z)$$

for a simple pole at z_0.

If there are more than one simple pole within the contour C, this result is easy to generalize. Instead of redrawing the contour with a small loop about the single pole, do it for all N poles within the contour. The result is clearly

$$\oint_C g(z)dz = 2\pi i \sum_{k=1}^{N} \text{Res}[g(z_k)] \tag{13}$$

We refer to this as the Cauchy Residue Theorem. It is widely used in mathematical physics.

The usefulness of the Residue Theorem can be illustrated in many ways, but here is one important example. It is a warm-up to evaluating the integral (6.2.9) in *Modern Quantum Mechanics, 2nd Ed.* The exercise is to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{ika}}{q^2 - k^2}\ k\ dk = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{ika}}{q^2 - k^2 + i\epsilon}\ k\ dk \tag{14}$$

where k, a, q, and $\epsilon > 0$ are all real variables. We use the second version above because this moves the singularities at $k = \pm q$ off the real axis. To be sure, we could have moved off the real axis by using $-i\epsilon$ instead of $+i\epsilon$, and in fact, this would give us a different answer. A physical rationale is needed to justify one sign or the other. Leave that for a physics course.

We can evaluate (14) using contour integration by first allowing k to be complex and then noting that $e^{ikx} \to 0$ as $\text{Im}(k) \to +\infty$. Therefore (14) can be rewritten as an integral over a semicircular contour C that runs (counter clockwise) along the $\text{Re}(k)$ axis and closes as a semicircle in the $\text{Im}(k) > 0$ plane. Then for $\epsilon \to 0$, the integrand in (14) has poles at

$$k = \pm\sqrt{q^2 - i\epsilon} = \pm q \left(1 - i\frac{\epsilon}{q^2}\right) \Rightarrow \pm q \mp i\epsilon$$

where we redefine ϵ (with $q > 0$) so that it is still small and has the same sign.

The pole at $k = k_0 \equiv +q - i\epsilon$ does not matter to us, since it is outside the integration contour. However, the pole at $k = -k_0 = -q + i\epsilon$ is inside, so we use the Residue Theorem to write

$$\begin{aligned} I &= \lim_{\epsilon \to 0} \oint_C \frac{e^{ika}}{(k - k_0)(k + k_0)}\, k\, dk \\ &= \lim_{\epsilon \to 0} 2\pi i\, \frac{e^{ika}}{k - k_0}\, k \bigg|_{k=-k_0} = \pi i \lim_{\epsilon \to 0} e^{-ik_0 a} = \pi i e^{-iqa} \end{aligned}$$

Other Topics

There is of course much more to complex analysis than what is covered here, even leaving aside the question of rigor. Two upper level texts I generally recommend are Arfken, G.B. and H.J. Weber. *Mathematical Methods for Physicists*, 4th ed., Academic Press, 1995; and Byron, F.W. and R.W. Fuller. *Mathematics of Classical and Quantum Physics*, Dover, 1992.

Some other important topics include the following:

- Convergence criteria
- Branch cuts in the complex plane
- The Cauchy Principal Value
- Conformal mapping