Hamiltonian for a Charged Particle in an Electromagnetic Field

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We will often need to know the Hamiltonian for a particle with mass m and charge q moving while in the presence of a static electric field $\mathbf{E}(\mathbf{x})$ and static magnetic field $\mathbf{B}(\mathbf{x})$. We know that the equation of motion is given by the Lorentz force law, namely

$$m\ddot{\mathbf{x}} = q\mathbf{E} + \frac{1}{c}q\dot{\mathbf{x}} \times \mathbf{B} \tag{1}$$

We can derive the Hamiltonian from the Lagrangian, but it is not obvious how to build the Lagrangian when there is no obvious "potential energy" function for a charged particle in a magnetic field.

Our approach[1] will be to start with the conjecture that the correct Lagrangian is given by

$$\mathscr{L}(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi(\mathbf{x}) + \frac{q}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) \tag{2}$$

where ϕ and $\mathbf{A}$ are the standard electrostatic and magnetic vector potentials, that is

$$\mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla}\phi(\mathbf{x}) \qquad \text{and} \qquad \mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x}), \tag{3}$$

and then show that Lagrange's Equations lead us to (1). We have

$$\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}_i} - \frac{\partial \mathscr{L}}{\partial x_i} \\
&= \frac{d}{dt}\left[m\dot{x}_i + \frac{q}{c}A_i(\mathbf{x})\right] + q\frac{\partial \phi}{\partial x_i} - \frac{q}{c}\dot{\mathbf{x}} \cdot \frac{\partial \mathbf{A}}{\partial x_i} \\
&= m(\ddot{\mathbf{x}})_i + q(\boldsymbol{\nabla}\phi)_i + \frac{q}{c}\left[\frac{d}{dt}A_i(\mathbf{x}) - \dot{\mathbf{x}} \cdot \frac{\partial \mathbf{A}}{\partial x_i}\right]
\end{aligned} \tag{4}$$

The first time on the right of (4) is just the left hand side of (1), and the second term on the right of (4) is just the first term on the right hand side of (1). It therefore remains to evaluate

$$\dot{\mathbf{x}} \cdot \frac{\partial \mathbf{A}}{\partial x_i} - \frac{d}{dt}A_i(\mathbf{x}) = \sum_j \dot{x}_j \frac{\partial A_j}{\partial x_i} - \sum_j \frac{\partial A_i}{\partial x_j}\dot{x}_j = \sum_j \dot{x}_j \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) \tag{5}$$

and show that it equals $(\dot{\mathbf{x}} \times \mathbf{B})_i = (\dot{\mathbf{x}} \times (\boldsymbol{\nabla} \times \mathbf{A}))_i$. We can use the totally antisymmetric symbol ε_{ijk} to write the cross product of two vectors as $(\mathbf{a} \times \mathbf{b})_i = \sum_j \sum_k \varepsilon_{ijk} a_k b_k$. Therefore

$$(\dot{\mathbf{x}} \times \mathbf{B})_i = \sum_j \sum_k \varepsilon_{ijk} \dot{x}_j \left(\sum_l \sum_m \varepsilon_{klm} \frac{\partial A_l}{\partial x_m}\right) = \sum_j \dot{x}_j \sum_l \sum_m \sum_k \varepsilon_{kij} \varepsilon_{klm} \frac{\partial A_l}{\partial x_m} \tag{6}$$

where $\varepsilon_{ijk} = \varepsilon_{kij}$ because the indices are rearranged by an even number of exchanges. We then make use of the theorem $\sum_k \varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ to write

$$(\dot{\mathbf{x}} \times \mathbf{B})_i = \sum_j \dot{x}_j \sum_l \sum_m (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial A_l}{\partial x_m} = \sum_j \dot{x}_j \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) \tag{7}$$

[1] One can derive this Lagrangian from first principles of relativity and electromagnetism, but we will leave that approach to an advanced course on classical or quantum field theory.

which is the same as (5). This proves our conjecture that (2) is the correct Lagrangian.

We can now use definitions from classical mechanics to derive the Hamiltonian $\mathscr{H}(\mathbf{p},\mathbf{x})$ from (2). First we determine the canonical momentum $\mathbf{p}$ from

$$p_i \equiv \frac{\partial \mathscr{L}}{\partial \dot{x}_i} = m\dot{x}_i + \frac{q}{c}A_i(\mathbf{x}) \tag{8}$$

Then we construct the Hamiltonian using the Legendre transformation

$$\begin{aligned}
\mathscr{H} &= \sum_i \dot{x}_i p_i - \mathscr{L} \\
&= \frac{1}{m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)\cdot\mathbf{p} - \frac{1}{2}m\frac{1}{m^2}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\phi - \frac{q}{c}\frac{1}{m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)\cdot\mathbf{A} \\
&= \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\phi
\end{aligned} \tag{9}$$

It is important to note that the canonical momentum $\mathbf{p}$ is not equal to $m\dot{\mathbf{x}}$ in this case.