PHYS5701 Quantum Mechanics I Spring 2018 HW #7

Due at 5pm to the **Instructor** on Wednesday 2 May 2018

You are to work independently on this problem set. You are free to use whatever notes, books, computers, or other reference works you feel are useful. You are also free to consult the course instructor or teaching assistant for help. You may *not*, however, consult other students in the class.

Please attach this page to your homework solutions along with your signature, below.

"I have complied with the requirement that I work independently on this problem set. I have not consulted with anyone other than the course instructor and teaching assistant in preparing these solutions."

Signature: ______________________________

Print name: ______________________________

Please note that there are a total of seven problems. All will be equally weighted.

HW #7 (Seven Problems)

(1) Using Equations (1.4.17) and (1.4.18), explicitly calculate **(a)** $S_x|S_x;+\rangle$, **(b)** $[S_x, S_y]$, and **(c)** $\langle S_x;+|S_y|S_x;+\rangle$, and explain why your answers are what you would expect.

(2) Prove the results (1.7.37), (1.7.38), (1.7.39), (1.7.40), and (1.7.41) using the wave function specified in (1.7.35).

(3)A particle of mass m moves in a one-dimension x under a potential energy function $V(x)$.

(a) For $V(x) = -V_0\, b\, \delta(x)$, $V_0 > 0$, $b > 0$, find the (one) bound state energy eigenvalue E.

(b) Generalize this to the "double delta function" potential

$$V(x) = -V_0 \frac{b}{2}\left[\delta\left(x + \frac{a}{2}\right) + \delta\left(x - \frac{a}{2}\right)\right]$$

and find the bound state energy eigenvalues and plot the corresponding eigenfunctions. Also show that you get the expected results as $a \to 0$.

(4) Consider a one-dimensional simple harmonic oscillator with frequency ω and eigenstates $|0\rangle$, $|1\rangle$, $|2\rangle$,.... A mixed ensemble is formed with equal parts of each of the three states

$$|\alpha\rangle \equiv \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right], \qquad |\beta\rangle \equiv \frac{1}{\sqrt{2}}\left[|1\rangle + |2\rangle\right], \qquad \text{and} \qquad |2\rangle$$

Find the density operator ρ and calculate the ensemble average of the energy.

(5) The "spin-angular functions" (aka "spinor spherical harmonics") are defined as

$$\mathcal{Y}_l^{j=l\pm1/2,m} = \frac{1}{\sqrt{2l+1}}\begin{bmatrix} \pm\sqrt{l \pm m + \frac{1}{2}}Y_l^{m-1/2}(\theta,\phi) \\ \sqrt{l \mp m + \frac{1}{2}}Y_l^{m+1/2}(\theta,\phi) \end{bmatrix}$$

See (3.8.64) in the textbook. These were constructed to be properly normalized eigenfunctions of $\mathbf{L}^2$, $\mathbf{S}^2$, $\mathbf{J}^2$, and J_z, where $\mathbf{J} \equiv \mathbf{L}+\mathbf{S}$. Use explicit calculations to prove the following:

(a) The $\mathcal{Y}_l^{j=l\pm1/2,m}$ are normalized, that is $\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \left(\mathcal{Y}_l^{j=l\pm1/2,m}\right)^\dagger \mathcal{Y}_l^{j=l\pm1/2,m} = 1$

(b) The $\mathcal{Y}_l^{j=l\pm1/2,m}$ have the correct eigenvalues for $\mathbf{J}^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2L_zS_z + L_+S_- + L_-S_+$, that is see (3.8.29), and J_z.

(6) Two spin-1 states are added to create a state $|j,m\rangle$. In a notation where $|j_{1,2}m_{1,2}\rangle = |1,\pm1\rangle$ or $|1,0\rangle$ are written $|\pm\rangle$ and $|0\rangle$, respectively

(a) Argue that $|2,2\rangle = |{+}{+}\rangle$ and then derive the states $|2,m\rangle$ for $m = 1, 0, -1, -2$. Check your answers by looking up the appropriate Clebsch-Gordan coefficients.

(b) Find the state $|1,1\rangle$ as a linear combination of $|{+}0\rangle$ and $|0{+}\rangle$ by enforcing orthogonality with $|2,1\rangle$. Then determine the states $|1,0\rangle$ and $|1,-1\rangle$, and check your answers .

(b) Finally, determine the state $|jm\rangle = |0,0\rangle$ and, again, check your answer.

(7) Using the defining property (3.11.8) for a vector operator, prove that the momentum operator $\mathbf{p}$ is a vector, based on its commutation relations with angular momentum $\mathbf{L}$.