PHYS5701 Quantum Mechanics I Spring 2018 HW #2

Due at 5pm to the **Grader** on Thursday 8 Feb 2018

(1) Starting with a momentum operator $\mathbf{p}$ having eigenstates $|\mathbf{p}'\rangle$, define an infinitesimal *boost* operator $\mathcal{B}(d\mathbf{p}')$ that changes one momentum eigenstate into another, that is

$$\mathcal{B}(d\mathbf{p}')|\mathbf{p}'\rangle = |\mathbf{p}' + d\mathbf{p}'\rangle$$

Show that the form $\mathcal{B}(d\mathbf{p}') = 1 + i\mathbf{W}\cdot d\mathbf{p}'$, where $\mathbf{W}$ is Hermitian, satisfies the unitary, associative, and inverse properties that are appropriate for $\mathcal{B}(d\mathbf{p}')$. Use dimensional analysis to express $\mathbf{W}$ in terms of the position operator $\mathbf{x}$, and show that the result satisfies the canonical commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$. Derive an expression for the matrix element $\langle\mathbf{p}'|\mathbf{x}|\alpha\rangle$ in terms of a derivative with respect to $\mathbf{p}'$ of $\langle\mathbf{p}'|\alpha\rangle$.

(2) For a wave function $\langle x'|\alpha\rangle = A(x'-a)^2(x'+a)^2e^{ikx'}$ for $-a \le x' \le a$ and zero otherwise,

(a) Find the constant A.
(b) Find the expectation values $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, and $\langle p^2\rangle$.
(c) Find the expectation values $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$, and compare their product to that for a Gaussian wave packet.

(3) Construct the unitary transformation U, as a sum over outer products of kets with bras, that transforms the spin-1/2 $|S_z;\pm\rangle$ basis into the $|S_y;\pm\rangle$. Your result should be in terms of the states $|\pm\rangle = |S_z;\pm\rangle$. Write the matrix representation of U in the $|S_z;\pm\rangle$ and show that matrix multiplication gives you the correct transformation.

(4) An electron sits in a uniform static magnetic field B in the $\hat{\mathbf{z}}$-direction, and is prepared at time $t=0$ in an eigenstate of the spin projection operator $\mathbf{S}\cdot\hat{\mathbf{n}}$ where $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}\hat{\mathbf{x}} + \frac{1}{\sqrt{2}}\hat{\mathbf{z}}$. Assuming the initial state eigenvalue is $+\hbar/2$, find the expectation values $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$ as a function of time. Explain why the answers are what you expect physically.

(5) Derive Equation (2.1.65) in the textbook. Then use the figure on the right, taken from Phys. Rev. D 95 (2017) 072006 (showing a neutrino disappearance measurement, similar to that shown in Figure 2.2 in the textbook) to estimate the mixing angle θ and mass-square-difference Δm^2 for this oscillation mode. You are welcome to look up the paper to check your answer.