SI and CGS Units in Electromagnetism

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December 28, 2017

Two divergent systems of units established themselves over the course of the 20th century. One system, known as *SI* (from the French Le Système International d'Unités), is rooted in the laboratory. It gained favor in the engineering community and forms the basis for most undergraduate curricula. The other system, called *Gaussian*, is aesthetically cleaner and is much favored in the theoretical physics community. We use the *Gaussian* system in this book, as do most graduate level physics texts on Quantum Mechanics and other subjects.

The *SI* system is also known as *MKSA* (for meter, kilogram, second, Ampere), and the *Gaussian* system is sometimes called[1] *CGS* (for centimeter, gram, second). For problems in mechanics, the difference is trivial, amounting only to some powers of ten. The difficulty comes when incorporating electromagnetism. As discussed below, *SI* incorporates a fourth base unit, the Ampere, which implies that charge and other electromagnetic quantities are dimensionally distinct between the SI and Gaussian systems. This one point is the source of all the confusion.

In other words, although we can write that a meter is equal to 100 centimeters, the *SI* unit of charge, called the Coulomb, is not equal to any number of electrostatic units (esu), the *Gaussian* unit of charge. The two kinds of charges literally have different physical meanings. They should probably have different names, but historically they are both called "charge." Similar comments apply to electric current, electric and magnetic fields, the electric potential, and so on. This why you see factors like ε_0 and μ_0 appear in *SI* formulas, whereas factors of c are common in *Gaussian* formulas.

This appendix gives an explanation for how these two systems of units diverge when it comes to electromagnetism. It also shows how physical quantities such as force and energy can be used to relate electromagnetic quantities between the two systems. For a more detailed discussion, see "On Electric and Magnetic Units and Dimensions", by R. T. Birge, *The American Physics Teacher*, 2(1934)41. (This journal is now called the *American Journal of Physics*.) See also the articles in the issue *American Journal of Physics* 3(2005) on pages 90, 102, and 171.

Electricity, Magnetism, and Electromagnetism: A Review

Electricity and magnetism are not separate phenomena. They are different manifestations of the same phenomenon, called electromagnetism. You need to incorporate special relativity to see how electricity and magnetism are united, and there were some decades between Maxwell and Einstein. Consequently, it was quite some time after they were separately established, that electricity and magnetism were realized to be just different ways that electromagnetism can exert a force.

The starting place for an "electric" force is Coulomb's Law. If some number of electrons is added to, or removed from, an object, then it acquires a "charge" q. A force appears between two charged objects separated by some distance. This force is proportional to the product of the charges, and inversely proportional to the square of the distance between them. That is,

$$F = k_E \frac{q_1 q_2}{d^2} \tag{1}$$

Here k_E is an arbitrary constant of proportionality; without describing what we mean by "charge," we can say no more about it.

[1] Some authors distinguish between *Gaussian* and *CGS* by including a factor of 4π in Gauss' Law.

A “magnetic” force appears between two wires, each of which carries something called a “current.” For two long, parallel wires, the force per unit length is proportional to the product of the currents and inversely proportional to the perpendicular separation of the wires. That is,

$$\frac{F}{L} = k_M \frac{I_1 I_2}{d} \tag{2}$$

As with the electric force, k_M is a generic constant of proportionality which depends on what we mean by “current.”

Today we understand that (1) and (2) are two different manifestations of “electromagnetism.” A “current” is in fact a flow of “charge,” and the theory of electromagnetism tells us that

$$2k_E = c^2 k_M \tag{3}$$

In other words, if we make a choice for k_E, then (3) specifies k_M and vice versa.

The essential point is that *the SI and Gaussian systems make different choices for k_E or k_M.* Other choices will lead to other systems of units, but we won’t be discussing them here. There are also variations on whether or not to combine factors of 4π into various quantities, but we are not going to distinguish between them in this appendix.

The SI System: Inventing a Unit for Current

The *SI* system is based on (2). People learned how to make current, well before we understood it in terms of charge. Perhaps for these reasons, a new base unit, the ampere (A), was created. One Ampere is the amount of current flowing in each of two long, parallel wires, separated by one meter, such that the force between the wires is 2×10^{-7} N/m. We write

$$k_M = \frac{\mu_0}{2\pi} \qquad \textbf{SI} \tag{4}$$

$$\text{where} \qquad \mu_0 \equiv 4\pi \times 10^{-7} \, \frac{\text{N}}{\text{A}^2} \tag{5}$$

(The factor of 4π turns out to be handy to cancel out integrations over the unit sphere.)

This quantity μ_0 turns out to describe the magnetic properties of the vacuum. It shows up, for example, in the inductance of a loop of wire surrounding empty space. This is all forced upon us by the invention of the Ampere. Some books refer to μ_0 as the “permeability of free space.”

Equations (3) and (4) tell us how to write Coulomb’s Law (1) in the *SI* system. We have

$$k_E = \frac{c^2}{2} k_M = \frac{\mu_0 c^2}{4\pi} \qquad \textbf{SI} \tag{6}$$

$$= 8.99 \times 10^9 \, \frac{\text{N} \cdot \text{m}^2}{(\text{A} \cdot \text{s})^2} \tag{7}$$

This form of Coulomb’s Law shows that charge, in the *SI* system, has units Amperes×seconds ($\text{A} \cdot \text{s}$). This is defined to be the Coulomb (C). *SI* furthermore defines the quantity

$$\varepsilon_0 \equiv \frac{1}{\mu_0 c^2} \tag{8}$$

called the “permittivity of free space.” It is another property of the vacuum, showing up, for example, as the capacitance of parallel plates separated by empty space. It is, as with μ_0, forced upon us by the choosing a new base unit for current. Thus, combining (1), (6), and (8), we write Coulomb’s Law in *SI* as

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{d^2} \qquad \textbf{SI} \tag{9}$$

which is the form presented in introductory physics textbooks that use the *SI* convention.

The Gaussian System: No New Base Units

In the *Gaussian* system, we take the point of view that no new base units are necessary. We write

$$k_E = 1 \qquad \textbf{Gaussian} \tag{10}$$

that is, a dimensionless number. In other words, Coloumb's Law (1) is simply

$$F = \frac{q_1 q_2}{d^2} \qquad \textbf{Gaussian} \tag{11}$$

The unit of charge in the *Gaussian* system is *derived* in terms of centimeters, grams, and seconds. It is called the electrostatic unit (esu), or sometimes the statcoulomb, and is simply[2]

$$\text{esu} \equiv \sqrt{\text{dyne}\cdot\text{cm}^2} = \text{g}^{1/2}\cdot\text{cm}^{3/2}/\text{s} \tag{12}$$

In this case, the magnetic force between wires is just (2) with (3), namely

$$\frac{F}{L} = \frac{2}{c^2}\frac{I_1 I_2}{d} \qquad \textbf{CGS} \tag{13}$$

The *Gaussian* system is not without its sources of confusion. Some authors use (13) to define a unit of current, the statampere, which gives 2 dynes/cm of force between two long parallel wires separated by 1 cm. Note that this *is not* the same as one esu/s, something called the "absolute ampere" or "abampere." The statampere and abampere differ by a factor of 2.998×10^{10}, although they have the same dimensions, namely those of the esu/s $= \text{g}^{1/2}\cdot\text{cm}^{3/2}/\text{s}^2$.

Converting between SI and CGS

It should now be clear to you that the units of charge and current have different *dimensions* between *SI* and *Gaussian*, and this is why everyone encounters confusion when converting between one system and the other.

Of course, all of this boils down to experiment. You make a measurement, and use some equations (whether they are *Gaussian* or *SI*) to interpret the result. We'll take the point of view of Coulomb's Law as a starting point, and the classic work by Millikan[3] to measure the charge on a single electron, a (negative) quantity that we traditionally call $-e$. The modern best value for his measurement is $e = 4.8032042\times 10^{-10}$ esu.

So let's start by giving ourselves the problem of expressing the charge on an electron in Coulombs. This is easy. Let e_{esu} equal the dimensionless number 4.8032042×10^{-10}. The force between two electrons separated by one *meter* is $10^{-4}e_{\text{esu}}^2$ dyne $= 10^{-9}e_{\text{esu}}^2$ N. So, in *SI*

$$10^{-9}e_{\text{esu}}^2\ \text{N} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{(1\ \text{m})^2} = \frac{\mu_0}{4\pi}\frac{c^2e^2}{(1\ \text{m})^2} = 10^{-7}c_{\text{SI}}^2 e_{\text{C}}^2\ \text{N} \tag{14}$$

where e_{C} is the electron charge in Coulombs, and $c_{\text{SI}} \equiv 2.998\times 10^8$, is yet another dimensionless number. Then $e_{\text{C}} = e_{\text{esu}}/10c_{\text{SI}} = 1.602\times 10^{-19}$. This procedure is obviously valid regardless of the charge on an electron. We therefore write

$$q_C = q_{\text{esu}}/10c_{\text{SI}} = q_{\text{esu}}/2.998\times 10^9 \tag{15}$$

as a general conversion between charge in the CGS system to that in SI. That is, one Coulomb represents a much larger amount of charge (i.e. very many more electrons) than one esu, by a factor of $10c_{\text{SI}}$.

The trick here was to recognize that the *numerical* difference between Coulombs and esu is absorbed by the factor c^2 in (14). This *is not* to say that Coulombs and esu differ by the dimensions of velocity. One *cannot* equate Coulombs to esu without some conversion factor that explicitly cancels out the base unit Amperes.

[2] Recall that the unit of force in CGS is called the dyne $\equiv \text{g}\cdot\text{cm}/\text{s}^2 = 10^{-5}$ N.

[3] This experiment was a tour de force, which Millikan carried out systematically and carefully over two decades. For a culmination of this work, see his paper "The Most Probable 1930 Values of the Electron and Related Constants" in Phys. Rev. 35(1930)1231. He determined the value $e = (4.770\pm 0.005)\times 10^{-10}$ esu.

We can extend from here. Consider the units of electric potential, defined by

$$\begin{aligned} 1 \text{ Joule} &= 1 \text{ Volt}\cdot\text{C} \qquad \textbf{SI} \\ 1 \text{ erg} &= 1 \text{ statvolt}\cdot\text{esu} \qquad \textbf{Gaussian} \\ \text{therefore} \qquad 1 \text{ Volt}\cdot\text{C} &= 10^7 \text{ statvolt}\cdot\text{esu} \end{aligned} \tag{16}$$

since one Joule equals 10^7 ergs. (I like this way of writing things because it means I can use the "=" sign. Energy is energy, whether *Gaussian* or *SI*.) Now thinking again in terms of number of electrons, we know that one Coulomb corresponds to $10c_{\text{SI}}$ times as much charge as an esu. So we write, now having to abandon a strict equality,

$$1 \text{ Volt}\cdot 10c_{\text{SI}} \iff 10^7 \text{ statvolt} \tag{17}$$

$$\text{or} \qquad 1 \text{ statvolt} \iff 299.8 \text{ Volt} \tag{18}$$

In other words, in practical terms, one statvolt is the same as 300 volts. Perhaps here is a reason that *SI* is more popular with electricians and engineers. We always prefer to use numbers on the order of unity when doing practical work. One volt is a reasonable potential difference from a human perspective, but 300 volts would give you a rather significant shock. So, if we worked in *Gaussian*, practical electronics would be discussed in terms of "millistatvolts", a somewhat unwieldy term.

The same thing works symbolically, of course. To go from *Gaussian* to *SI* we need to insert the factor $\mu_0 c^2/4\pi = 1/4\pi\varepsilon_0$ in front of Coulomb's law, and rederive things. So, wherever we encounter a value of charge q in a *Gaussian* equation, we replace it with $q/\sqrt{4\pi\varepsilon_0}$. (The trivial conversions from centimeters and grams to meters and kilograms are irrelevant symbolically.) Similarly, any values of current I get replaced by $cI\sqrt{\mu_0/4\pi}$.

You can easily check that these substitutions turn (11) into (9), and (13) into (2) w/(4). For a different example, the electric field $\mathbf{E} = \lim_{q_0\to 0}\mathbf{F}/q_0$ gets multiplied by $\sqrt{4\pi\varepsilon_0}$ leaving (of course) the force for a charge in an electric field $\mathbf{F} = q\mathbf{E}$ unchanged.

Now let's try going the other way, namely *SI* to *Gaussian*, with the Lorentz force law, namely

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B} \qquad \textbf{SI} \tag{19}$$

We know that the first term on the right is unchanged. In the second term, replace q with $q\sqrt{4\pi\varepsilon_0} = q\sqrt{4\pi/\mu_0}/c$, but what about the $\mathbf{B}$ field? A study of electromagnetism leads to Ampere's Law, which relates magnetic fields and currents. Indeed, steady currents give rise to static magnetic fields according to

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} \qquad \textbf{Gaussian} \tag{20a}$$

$$\nabla\times\mathbf{B} = \mu_0\,\mathbf{j} \qquad \textbf{SI} \tag{20b}$$

where $\mathbf{j}$ is the current density. To go from CGS to SI, $\mathbf{j}$ gets multiplied by $c\sqrt{\mu_0/4\pi}$, so equations (20) tell us to multiply $\mathbf{B}$ by $\sqrt{4\pi/\mu_0}$. So, getting back to (19) we multiply the second term by $\sqrt{4\pi/\mu_0}/c$ for q and $\sqrt{\mu_0/4\pi}$ for $\mathbf{B}$. (Remember, we are going from SI to CGS.) We therefore arrive at the Lorentz force law

$$\mathbf{F} = q\mathbf{E} + q\frac{\mathbf{v}}{c}\times\mathbf{B} \qquad \textbf{Gaussian} \tag{21}$$

An important lesson in this last example is that in going from *SI* to *Gaussian*, the magnetic field changes its dimensions differently than does the electric field. Indeed, (19) shows that in *SI*, the dimensions of $\mathbf{E}$ are the dimensions of $\mathbf{B}$ multiplied by the dimensions of velocity. On the other hand, (21) shows that in the *Gaussian* system, $\mathbf{E}$ and $\mathbf{B}$ have the *same* dimensions. Pedagogically, this is an advantage that the *Gaussian* system has over *SI*.

For a final example, let's express the magnetic moment in *Gaussian* units, which is the starting point for this book. In your introductory physics class, which most likely used *SI* units, you defined the magnetic moment $\mu = I\mathscr{A}$ for a current I moving in a closed loop that enclosed an area $\mathscr{A}$. You also learned that the potential energy for such a current loop in a magnetic field $\mathbf{B}$ was $-\boldsymbol{\mu}\cdot\mathbf{B}$, which of course must be the same expression as in the *Gaussian* system. We already know that $I \to c\sqrt{\mu_0/4\pi}I$ and $\mathbf{B} \to \sqrt{4\pi/\mu_0}\mathbf{B}$. Therefore, in order to keep the same expression for potential energy, the definition of magnetic moment in *Gaussian* units must be $\mu = I\mathscr{A}/c$.