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Eigenvalues + Eigenvectors of Angular Momentum

Three operators J_x, J_y, J_z

$$[J_x, J_y] \equiv J_x J_y - J_y J_x = i\hbar J_z$$

$$[J_z, J_x] = i\hbar J_y \qquad [J_y, J_z] = i\hbar J_x$$

Example: S_x, S_y, S_z

w/ $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$

Define $\vec{J}^2 \equiv J_x^2 + J_y^2 + J_z^2$

$$[\vec{J}^2, J_z] = [J_x^2, J_z] + [J_y^2, J_z] + 0$$

$$[J_x^2, J_z] = J_x^2 J_z - J_x J_z J_x + J_x J_z J_x - J_z J_x^2$$

$$= J_x [J_x, J_z] + [J_x, J_z] J_x$$

$$= -i\hbar J_x J_y - i\hbar J_y J_x$$

$$[J_y^2, J_z] = +i\hbar J_y J_x + i\hbar J_x J_y$$

$+ = 0$

So $\boxed{[\vec{J}^2, J_z] = 0}$

"Simultaneous Eigenstates"

$$\vec{J}^2 |a,b\rangle = a|a,b\rangle$$

$$J_z |a,b\rangle = b|a,b\rangle$$

"Ladder Operators" $\quad J_\pm = J_x \pm i J_y$

- $\boxed{J_\pm^\dagger = J_\mp}$
- $\boxed{[\vec{J}^2, J_\pm] = 0}$
- $[J_z, J_\pm] = [J_z, J_x] \pm i [J_z, J_y]$

$$= i\hbar J_y \pm i(-i\hbar J_x)$$

$$= \pm\hbar(J_x \pm i J_y) = \pm\hbar J_\pm$$

What does $J_\pm$ do to $|a,b\rangle$?

(1) $\vec{J}^2 [J_\pm |a,b\rangle] = J_\pm \vec{J}^2 |a,b\rangle = a[J_\pm |a,b\rangle]$

(2) $J_z [J_\pm |a,b\rangle]$

$$= (J_\pm J_z + [J_z, J_\pm])|a,b\rangle$$

$$= (b J_\pm \pm \hbar J_\pm)|a,b\rangle$$

$$= (b \pm \hbar)[J_\pm |a,b\rangle]$$

$$\Rightarrow J_\pm |a,b\rangle = C_\pm |a, b\pm\hbar\rangle$$

$$J_+ J_- = (J_x + iJ_y)(J_x - iJ_y)$$
$$= J_x^2 + J_y^2 + i(J_y J_x - J_x J_y)$$
$$\boxed{J_- J_+ = J_x^2 + J_y^2 - i(J_y J_x - J_x J_y)}$$
$$J_x^2 + J_y^2 = \boxed{\vec{J}^2 - J_z^2 = \frac{1}{2}(J_+ J_- + J_- J_+)}$$
$$= \frac{1}{2}(J_-^\dagger J_- + J_+^\dagger J_+)$$
$$\langle a,b|(\vec{J}^2 - J_z^2)|a,b\rangle = a - b^2$$
$$= \frac{1}{2}\langle a,b|[J_-^\dagger J_- + J_+^\dagger J_+]|a,b\rangle$$

But $\langle\alpha|X^\dagger X|\alpha\rangle = [\langle\alpha|X^\dagger][X|\alpha\rangle] \geq 0$

$\Rightarrow$ $a - b^2 \geq 0$ or $\boxed{a \geq b^2}$

Recall $J_\pm |a,b\rangle = C_\pm |a, b \pm \hbar\rangle$

$\Rightarrow$ $J_+ |a, b_{max}\rangle = 0$ for $b = b_{max}$

$J_- |a, b_{min}\rangle = 0$ for $b = b_{min}$

Use $J_- J_+ |a, b_{max}\rangle = 0$

$J_+ J_- |a, b_{min}\rangle = 0$

But $J_- J_+ = J_x^2 + J_y^2 + i\,[J_x, J_y]$

$$= \vec{J}^2 - J_z^2 - \hbar J_z$$

$$\Rightarrow \quad a - b_{max}^2 - \hbar b_{max} = 0$$

i.e. $\boxed{a = b_{max}^2 + \hbar b_{max}}$ ⟸

Also $J_+ J_- = \vec{J}^2 - J_z^2 - i\,[J_x, J_y]$

$$\Rightarrow \quad \boxed{a = b_{min}^2 - \hbar b_{min}}$$

$$b_{max}^2 + \hbar b_{max} - b_{min}^2 + \hbar b_{min} = 0$$

$$b_{max} = \frac{1}{2}\left[-\hbar \pm \left(\underbrace{\hbar^2 - 4\hbar b_{min} + 4 b_{min}^2}_{(\hbar - 2b_{min})^2}\right)^{1/2}\right]$$

$$= \frac{1}{2}\left[-\hbar - \hbar + 2b_{min}\right] = b_{min} - \hbar \quad \times$$

or $$= \frac{1}{2}\left[-\hbar + \hbar - 2b_{min}\right] = -b_{min} \quad \checkmark$$

$$\Rightarrow \quad b_{max} - b_{min} = \underline{\text{integer}}$$

$$= (0, 1, 2, 3, \ldots)\,\hbar$$

$$= 2b_{max} \Rightarrow b_{max} = 0, \hbar/2, \hbar, 3\hbar/2, \ldots$$

Rename Eigenvalues

$$b_{max} = j\hbar \qquad j = \frac{\text{Positive Integer}}{2}$$

$$= 0,\ 1/2,\ 1,\ 3/2,\ \ldots$$

$$a = j^2\hbar^2 + j\hbar^2 = j(j+1)\hbar^2$$

$$b \equiv m\hbar \qquad m = -j, -j+1, \ldots, j-1, j$$

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we have discovered that

$$\boxed{\begin{gathered} \vec{J}^2 \, |j\,m\rangle = j(j+1)\hbar^2 \, |j\,m\rangle \\ \underline{\underline{J_z}} \, |j\,m\rangle = \underline{\underline{m}}\hbar \, |j\,m\rangle \\ j = 0,\ 1/2,\ 1,\ 3/2,\ 2,\ \ldots \\ m = -j, -j+1, \ldots, j-1, j \rightarrow \underline{\underline{2j+1}} \end{gathered}}$$

Recall: $S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$

i.e. $j = 1/2 \Rightarrow$ "Spin-$\underline{\underline{1/2}}$"

$= j$

Matrix Elements

$$\langle j'm' | \vec{J}^2 | jm \rangle = j(j+1)\hbar^2 \, \delta_{j'j} \, \delta_{m'm}$$

$$\langle j'm' | J_z | jm \rangle = m\hbar \, \delta_{j'j} \, \delta_{m'm}$$

Recall $J_\pm |j,m\rangle = c_\pm |j, m\pm 1\rangle$

$$\left[\langle j,m| J_+^\dagger\right]\left[J_+ |jm\rangle\right] = |c_+|^2$$

$$= \langle j,m | J_- J_+ | jm \rangle$$

$$= \langle jm | \left[\vec{J}^2 - J_z^2 - \hbar J_z \right] | jm \rangle$$

$$\Rightarrow |c_+|^2 = j(j+1)\hbar^2 - m^2\hbar^2 - m\hbar^2$$

$$= \left[j^2 - m^2 + j - m \right] \hbar^2$$

$$= \left[(j-m)(j+m) + (j-m) \right] \hbar^2$$

$$\Rightarrow c_+ = \left[(j-m)(j+m+1) \right]^{1/2} \hbar$$

$$c_- = \left[(j+m)(j-m+1) \right]^{1/2} \hbar$$

$$\langle j'm' | J_\pm | jm \rangle$$

$$= \left[(j \mp m)(j \pm m + 1) \right]^{1/2} \hbar \, \delta_{j'j} \, \delta_{m', m\pm 1}$$

Our New "Basis" is Spin-j!

$\Leftrightarrow$ Represent operators in $|jm\rangle$

<u>Example</u>: Rotation operators

$$\mathcal{D}^{(j)}_{m'm}(R) = \langle jm' | \mathcal{D}(R) | jm \rangle$$

Matrix!

"Wigner Function"