Phys 5701 Quantum Mech I

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From Last Class

$$\left.\begin{array}{ll} & |S_z;+\rangle \text{ and } |S_z;-\rangle \\ \text{OR} & |S_x;+\rangle \text{ and } |S_x;-\rangle \\ \text{OR} & |S_y;+\rangle \text{ and } |S_y;-\rangle \end{array}\right\} \begin{array}{l} \text{w/ } S_z \\ (\text{or } S_x, S_y) \\ = \pm\frac{\hbar}{2} \end{array}$$

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Kets, Bras, Operators

$|\alpha\rangle$ "ket"

$\alpha$ = "whatever"

$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$

$c|\alpha\rangle = |\alpha\rangle c \quad c$ = complex number

and $|\alpha\rangle \longrightarrow$ Same state!

If $c = 0 \Longrightarrow$ "Null ket"

## The Dual Space

$$|\alpha\rangle \xleftrightarrow[\text{DC}]{\text{"Dual correspondence"}} \langle\alpha| \quad \text{"bra"}$$

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle \longleftrightarrow \langle\alpha| + \langle\beta| = \langle\gamma|$$

**Postulate**: $c\,|\alpha\rangle \xleftrightarrow{\text{DC}} c^*\langle\alpha|$

$$\Leftrightarrow \quad c_\alpha|\alpha\rangle + c_\beta|\beta\rangle \xleftrightarrow{\text{DC}} c_\alpha^*\langle\alpha| + c_\beta^*\langle\beta|$$

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## Inner Product

$$\underset{\text{"bra"}}{(\langle\beta|)} \cdot \underset{\text{"ket"}}{(|\alpha\rangle)} = \underset{\text{"bra[c]ket"}}{\langle\beta|\alpha\rangle}$$

complex number

**Postulate**: $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$

__Postulate__: "Positivity Postulate"

$$\langle\alpha|\alpha\rangle \geq 0 \quad (= 0 \text{ only if } |\alpha\rangle = \text{null ket})$$

"Positive Real Number"

Terminology:

If $\langle\alpha|\alpha\rangle = 1$ "NORMALIZED"

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However $c|\alpha\rangle$ is the same state!

$\Rightarrow$ $\dfrac{1}{\sqrt{\langle\alpha|\alpha\rangle}}|\alpha\rangle$ is __normalized__

$\langle\alpha|\beta\rangle = 0 \Rightarrow$ "orthogonal"

__also__ $|\alpha_1\rangle, |\alpha_2\rangle, |\alpha_3\rangle, \ldots$

normalized, mutually orthogonal

$\Rightarrow$ "orthonormal" set

# Operators

$$X \cdot (|\alpha\rangle) = X|\alpha\rangle \qquad \cancel{|\alpha\rangle X}$$

$$X \cdot (\langle\beta|) = \langle\beta| X$$

NOT SAME STATE AS $|\alpha\rangle$

$X = Y$ IFF $X|\alpha\rangle = Y|\alpha\rangle$ FOR ANY $|\alpha\rangle$

$X|\alpha\rangle = 0 \qquad X =$ "Null operator"

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Addition is commutative
distributive
associative

$$X + Y = Y + X$$

$$X[c_\alpha|\alpha\rangle + c_\beta|\beta\rangle] = c_\alpha X|\alpha\rangle + c_\beta X|\beta\rangle$$

$$X + (Y + Z) = (X + Y) + Z$$

## Dual Space:

$$X|\alpha\rangle \overset{?}{\longleftrightarrow} \langle\alpha|X \quad \textbf{NO!}$$

$$X|\alpha\rangle \longleftrightarrow \langle\alpha|X^{+}$$

"ADJOINT OPERATOR"

If $X = X^{+}$

then $X =$ "Hermitian"

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Multiplication is associative

$$x(yz) = (xy)z$$

is not commutative!

$$xy \neq yx$$

also $\quad X(y|\alpha\rangle) = (xy)|\alpha\rangle = xy|\alpha\rangle$

etc...

**Theorem:** $(XY)^{+} = Y^{+}X^{+}$

**Proof:** $XY|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|(XY)^{+}$

Put $XY|\alpha\rangle = X(Y|\alpha\rangle)$

$$\xleftrightarrow{DC} (\langle\alpha|Y^{+})X^{+}$$

$$= \langle\alpha|Y^{+}X^{+}$$

PROVED!

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**Outer Products**

e.g. $|\beta\rangle\langle\alpha|$ **OPERATOR!**

i.e. $X \equiv |\beta\rangle\langle\alpha|$

$\hookrightarrow$ $X|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle$

**Theorem:** $X^{+} = |\alpha\rangle\langle\beta|$

$$(\langle\beta|)\cdot(X|\alpha\rangle) = \boxed{\langle\beta|X|\alpha\rangle}$$

$$= (\langle\beta|X)\cdot|\alpha\rangle$$

<u>theorem</u>: $\langle\beta|X|\alpha\rangle = \langle\alpha|X^{+}|\beta\rangle^{*}$

$$\langle\beta|X|\alpha\rangle = \langle\beta|(X|\alpha\rangle) = \left[(\langle\alpha|X^{+})(|\beta\rangle)\right]^{*}$$

$$= \langle\alpha|X^{+}|\beta\rangle^{*}$$

IF $X = X^{+} \Rightarrow \langle\beta|X|\alpha\rangle = \langle\alpha|X|\beta\rangle^{*}$

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<u>Physics</u>

"$|S_z;\pm\rangle$ are eigenstates of $S_z$"

$S_z$ = (Hermitian) operator

<u>Postulate</u>: All observables correspond to Hermitian operators!

$$S_z|S_z;\pm\rangle = \pm\frac{\hbar}{2}|S_z;\pm\rangle$$