Phys 5701   24 Sep 2020

Today: Simple Harmonic Oscillator

ONE DIMENSION: x, p

$$\boxed{H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 x^2}$$ i.e. "k" $= m\omega^2$

x & p are "canonically conjugate"

i.e. $[x,p] = i\hbar$

Goal: Solve $H|E\rangle = E|E\rangle$

Start: $$a \equiv \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x + i\frac{p}{m\omega}\right)$$

$$\Leftrightarrow \quad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x - i\frac{p}{m\omega}\right)$$

$$a^\dagger a = \frac{m\omega}{2\hbar}\left(x^2 + \frac{p^2}{m^2\omega^2} - \frac{i}{m\omega}px + \frac{i}{m\omega}xp\right)$$

But $xp - px = i\hbar$

$$a^\dagger a = \frac{m\omega}{2\hbar}\left(x^2 + \frac{1}{m^2\omega^2}p^2 + \frac{i}{m\omega}(i\hbar)\right)$$

$$= \frac{m\omega}{2\hbar}\left(x^2 + \frac{1}{m^2\omega^2}p^2\right) - \frac{1}{2}$$

$$= \frac{1}{\hbar\omega}\underbrace{\left(\frac{1}{2}m\omega^2x^2 + \frac{1}{2m}p^2\right)}_{H} - \frac{1}{2}$$

$$N \equiv a^\dagger a \Rightarrow \boxed{H = \hbar\omega\,(N + 1/2)}$$

$a =$ "Annihilation" operator

$a^\dagger =$ "Creation" operator

$N =$ "Number" operator

$$N^\dagger = (a^\dagger a)^\dagger = a^\dagger (a^\dagger)^\dagger = a^\dagger a = N$$

Hermitian!

$$\boxed{N|n\rangle = n|n\rangle} \qquad (A|a'\rangle = a'|a'\rangle)$$

$$H|n\rangle = E_n|n\rangle \qquad E_n = (n+1/2)\hbar\omega$$

$$[a,a^+] = \frac{m\omega}{2\hbar}[x+ip/m\omega \,,\, x-ip/m\omega]$$

$$= \frac{m\omega}{2\hbar}\Big[\cancel{x}^2 - i\frac{xp}{m\omega} + i\frac{px}{m\omega} + \frac{\cancel{p}^2}{m^2\omega^2}$$

$$-\cancel{x}^2 + i\frac{px}{m\omega} - i\frac{xp}{m\omega} - \frac{\cancel{p}^2}{m^2\omega^2}\Big]$$

$$= \frac{m\omega}{2\hbar}\Big[-2i\frac{xp}{m\omega} + 2i\frac{px}{m\omega}\Big]$$

$$= \frac{-i}{\hbar}\underline{(xp-px)} = 1$$

i.e. $[a,a^+] = \boxed{\boxed{aa^+ - a^+a = 1}}$

$$[N,a] = [a^\dagger a, a] = a^\dagger a a - a a^\dagger a$$
$$= (a^\dagger a - a a^\dagger) a = -a$$

$$[N, a^\dagger] = a^\dagger a a^\dagger - a^\dagger a^\dagger a$$
$$= a^\dagger (a a^\dagger - a^\dagger a) = +a^\dagger$$

i.e. $Na = aN - a$

$Na^\dagger = a^\dagger N + a^\dagger$

$$\boxed{N[a^\dagger |n\rangle]} = N a^\dagger |n\rangle$$
$$= (a^\dagger N + a^\dagger)|n\rangle$$
$$= \boxed{(n+1)[a^\dagger |n\rangle]}$$

$\Leftrightarrow$ $\boxed{a^\dagger |n\rangle = c\,|n+1\rangle}$ "Creation"

$$\langle n | a a^\dagger |n\rangle = \langle n | [a^\dagger a + 1] | n\rangle$$
$$= n+1$$

$$= [\langle n|a][a^\dagger|n\rangle] = c^* c \langle n+1|n+1\rangle$$
$$= |c|^2$$

Choose c Real.

$$\Rightarrow c = (n+1)^{1/2}$$

i.e. $a^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle$

also $a|n\rangle = n^{1/2}|n-1\rangle$

What is n?

$$\langle n|N|n\rangle = \boxed{n}$$
$$= [\langle n|a^\dagger][a|n\rangle] \boxed{\geq 0}$$

"Positivity Postulate"

$$a|n\rangle = n^{1/2}|n-1\rangle$$
$$a^2|n\rangle = n^{1/2}(n-1)^{1/2}|n-2\rangle$$

... Eventually $n < 0$!! Bad!!

UNLESS you reach $a|0\rangle = 0^{1/2}|0\rangle$
$= 0$!!

↳ n must be non-negative integer!!

i.e. $E_n = (n+\frac{1}{2})\hbar\omega$

$n = 0, 1, 2, 3, \ldots, \infty$

NOTE: $E_0 = \frac{1}{2}\hbar\omega \neq 0$

Matrix Elements

$\langle n'|N|n\rangle = n\,\delta_{n'n}$ $\quad N \doteq \begin{bmatrix} 0 & & & \\ & 1 & & 0 \\ & & 2 & \\ 0 & & & 3 \\ & & & & \ddots \end{bmatrix}$

$$\boxed{\begin{aligned} \langle n'|a|n\rangle &= n^{1/2}\,\delta_{n',n-1} \\ \langle n'|a^\dagger|n\rangle &= (n+1)^{1/2}\,\delta_{n',n+1} \end{aligned}}$$

Recall $\boxed{a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x + \frac{i}{m\omega}p\right)}$

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x - \frac{i}{m\omega}p\right)$$

$$\Leftrightarrow \boxed{\begin{aligned} x &= \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger) \\ p &= i\left(\frac{\hbar}{2m\omega}\right)^{1/2}(-a + a^\dagger) \end{aligned}}$$

so $\langle n'|x|n\rangle$ or $\langle n'|p|n\rangle$ are easy.

Wave Functions

$$a|0\rangle = 0 \Rightarrow \langle x'|a|0\rangle = 0$$

$$\langle x'|\left[x + \frac{i}{m\omega}p\right]|0\rangle = 0$$

$$x'\langle x'|0\rangle + \frac{i}{m\omega}\frac{\hbar}{i}\frac{d}{dx'}\langle x'|0\rangle = 0$$

i.e. $x\, u(x) + \frac{\hbar}{m\omega}\frac{du}{dx} = 0$

$$\frac{du}{u(x)} = -\frac{m\omega}{\hbar} x \qquad dx$$

$$\log u = -\frac{1}{2}\frac{m\omega}{\hbar} x^2 + c'$$

$$\Rightarrow u(x') = \boxed{\langle x'|0\rangle = c\, e^{-m\omega x'^2/2\hbar}}$$

But $\langle 0|0\rangle = 1 = \int dx' \langle 0|x'\rangle\langle x'|0\rangle$

$$= \int dx'\, c^2 e^{-m\omega x'^2/\hbar}$$

$$\Longrightarrow c = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

Then: $\langle x'|a^\dagger|0\rangle = \langle x'|1\rangle\, 1^{1/2}$

$$= \left(\frac{m\omega}{2\hbar}\right)^{1/2} \langle x'|\left[x - \frac{i}{m\omega} p\right]|0\rangle$$

$$\to \frac{\hbar}{i}\frac{d}{dx'}$$

Uncertainty Relation

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} \left| \langle [A,B] \rangle \right|^2$$

$$\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

Do this for $A \to x$

$B \to p$

$|\alpha\rangle = |n\rangle$

$$\langle x \rangle = 0 = \langle p \rangle$$

$$x^2 = \frac{\hbar}{2m\omega} \left(\cancel{a^2} + a a^\dagger + a^\dagger a + \cancel{a^{\dagger 2}} \right)$$

Take $|\alpha\rangle = |0\rangle$

$$\Rightarrow \langle x^2 \rangle = \frac{\hbar}{2m\omega}$$

also $\langle p^2 \rangle = \dfrac{\hbar m\omega}{2}$

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle = \frac{\hbar^2}{4}$$

$$\frac{1}{4} \left| \langle [x,p] \rangle \right|^2 = \frac{1}{4} |i\hbar|^2 = \frac{\hbar^2}{4}$$

MINIMUM UNCERTAINTY!

For $|\alpha\rangle = |n\rangle$

$$\langle (\Delta x)^2 \rangle \langle (\Delta p^2) \rangle = \left(n+\frac{1}{2}\right)^2 \hbar^2$$

<u>Time Development</u>: Homework

Heisenberg $\rightarrow$ $\frac{dx}{dt} = \frac{p}{m}$ $\quad \frac{dp}{dt} = -m\omega^2 x$

$$\Leftrightarrow \frac{da}{dt} = -i\omega a$$

i.e. $a(t) = e^{-i\omega t} a(0)$

also $a^\dagger(t) = e^{+i\omega t} a(0)$

"Coherent States": $a|\lambda\rangle = \lambda|\lambda\rangle$

$$|\lambda\rangle = \sum_n c_n |n\rangle$$