Phys 5701 22 Sep

From Last Class

$$|\alpha, t_0; t\rangle = \mathcal{U}(t,t_0)|\alpha, t_0\rangle \qquad t > t_0$$

$$\Rightarrow \boxed{i\hbar \frac{\partial}{\partial t}\mathcal{U}(t,t_0) = H\,\mathcal{U}(t,t_0)}$$

so $\quad i\hbar \dfrac{\partial}{\partial t}|\alpha, t_0; t\rangle = H|\alpha, t; t_0\rangle$

Project onto representation.
e.g. $|\pm\rangle$ or $|\vec{x}'\rangle$
$\Rightarrow$ "Normal" Differential Equations

But First: For H independent of time

$$\Rightarrow \mathcal{U}(t,t_0) = \exp\left[-\frac{i}{\hbar}H(t-t_0)\right]$$

$\Rightarrow$ Study evolution directly!

$$H|\pm\rangle = E_\pm|\pm\rangle \;\ldots$$

Today: Schrödinger Picture
vs Heisenberg Picture

Consider unitary operator $U|\alpha\rangle = |\beta\rangle$

$$\langle A\rangle_\beta = \langle\beta|A|\beta\rangle = [\langle\alpha|U^\dagger]\, A\, [U|\alpha\rangle]$$

$$= \langle\alpha|\left[U^\dagger A U\right]|\alpha\rangle$$

Two "pictures": (1) $|\alpha\rangle \to |\beta\rangle$
(2) $\boxed{A \to U^\dagger A U}$

Example: $\mathcal{J}(d\vec{x}')|\alpha\rangle = \left[1 - \frac{i}{\hbar}\vec{p}\cdot d\vec{x}'\right]|\alpha\rangle$

$$\mathcal{J}^\dagger(d\vec{x}')\,\vec{x}\,\mathcal{J}(d\vec{x}')$$

$$= \left[1 + \frac{i}{\hbar}\vec{p}\cdot d\vec{x}'\right]\vec{x}\left[1 - \frac{i}{\hbar}\vec{p}\cdot d\vec{x}'\right]$$

$$= \vec{x} + \frac{i}{\hbar}\left[\underline{(\vec{p}\cdot d\vec{x}')\,\vec{x}} - \vec{x}\,(\vec{p}\cdot d\vec{x}')\right]$$

But $(\vec{p}\cdot d\vec{x}')\,x_i - x_i(\vec{p}\cdot d\vec{x}')$

$$= p_j\,dx_j'\,x_i - x_i\,p_j\,dx_j'$$

$$= -\,(x_i p_j - p_j x_i)\,dx_j'$$

$$= -[x_i, p_j]\,dx_j' = -i\hbar\,\delta_{ij}\,dx_j' = -i\hbar\,dx_i'$$

$$\Rightarrow \boxed{\mathcal{J}^{+}(d\vec{x}')\,\vec{x}\,\mathcal{J}(d\vec{x}') = \vec{x} + \frac{i}{\hbar}\left(-i\hbar\, d\vec{x}'\right) = \vec{x} + d\vec{x}'\,\mathbb{1}}$$

Right!

Apply to $\mathcal{U}(t, t_0)$

Take $t_0 = 0 \Rightarrow \mathcal{U}(t) = e^{-iHt/\hbar}$

$$\boxed{(1)\quad |\alpha, t\rangle = \mathcal{U}(t)\,|\alpha\rangle \;;\; \langle A\rangle = \langle \alpha, t|\,A^{(S)}\,|\alpha, t\rangle}$$

"Schrödinger Picture"

(2) $\langle A\rangle = \langle \alpha|\,A^{(H)}\,|\alpha\rangle$

$$\boxed{A^{(H)} = \mathcal{U}^{+}(t)\,A^{(S)}\,\mathcal{U}(t)} = A^{(H)}(t)$$

"Heisenberg Picture"

$$\frac{d}{dt}A^{(H)}(t)$$ w/ all time dependence in $\mathcal{U}(t)$

$$= \left[\frac{\partial \mathcal{U}^{\dagger}(t)}{\partial t}\right] A^{(s)}\,\mathcal{U}(t) + \mathcal{U}^{\dagger}(t)\,A^{(s)}\,\frac{\partial \mathcal{U}(t)}{\partial t}$$

But $\frac{\partial \mathcal{U}}{\partial t} = \frac{1}{i\hbar} H\,\mathcal{U} \Rightarrow \frac{\partial \mathcal{U}^{\dagger}}{\partial t} = -\frac{1}{i\hbar}\,\mathcal{U}^{\dagger} H$

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so $$\frac{d}{dt}A^{(H)}(t) = \frac{1}{i\hbar}\left[\mathcal{U}^{\dagger}A^{(s)}H\,\mathcal{U} - \mathcal{U}^{\dagger}H A^{(s)}\mathcal{U}\right]$$

$$= \frac{1}{i\hbar}\left[\mathcal{U}^{\dagger}A^{(s)}\mathcal{U}\,H - H\,\mathcal{U}^{\dagger}A^{(s)}\mathcal{U}\right]$$

$$= \frac{1}{i\hbar}\left[A^{(H)}H - H\,A^{(H)}\right]$$

so $$\boxed{\frac{d}{dt}A(t) = \frac{1}{i\hbar}[A,H]}$$

in Heisenberg Picture.

note: $\mathcal{U}^{\dagger}H\,\mathcal{U} = H\,\underline{\underline{\mathcal{U}^{\dagger}\mathcal{U}}} = H$

"Heisenberg Equation of Motion"

Application: Free Particles

$$H = \frac{1}{2m}\vec{p}^{\,2} = \frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2\right]$$

$$= \frac{1}{2m} p_j p_j$$

momentum

$$\frac{dp_i}{dt} = \frac{1}{i\hbar}[p_i, H] = 0$$

$$\boxed{[x, p] = i\hbar}$$

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position

$$\frac{dx_i}{dt} = \frac{1}{i\hbar}[x_i, H] = \frac{1}{i\hbar}\frac{1}{2m}[x_i, p_j p_j]$$

$$[x_i, p_j p_j] = x_i p_j p_j - p_j p_j x_i$$

$$= x_i p_j p_j - p_j x_i p_j + p_j x_i p_j - p_j p_j x_i$$

$$= [x_i, p_j] p_j + p_j [x_i, p_j]$$

$$= i\hbar \delta_{ij} p_j + p_j i\hbar \delta_{ij}$$

$$= 2i\hbar p_i$$

$$\frac{dx_i}{dt} = \frac{1}{i\hbar}\frac{1}{2m} 2i\hbar p_i = \frac{p_i}{m}$$

i.e. $\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m}$ <u>or</u> $\vec{p} = m\frac{d\vec{x}}{dt}$ <u>Right!</u>

I calculated $[x_i, \vec{p}^{\,2}] = 2i\hbar p_i$

<u>But</u> $[x_i, F(\vec{p})] = i\hbar \frac{\partial F}{\partial p_i}$ (2.97a)

and $[p_i, G(\vec{x})] = -i\hbar \frac{\partial G}{\partial x_i}$ (2.97b)

Now $\quad H = \frac{1}{2m}\vec{p}^{\,2} + V(\vec{x}) \quad = \frac{1}{2}m\omega^2\vec{x}^{\,2}$

Still $\frac{d\vec{x}}{dt} = \frac{1}{m}\vec{p}$ because $[x_i, V(\vec{x})] = 0$

$$\frac{dp_i}{dt} = \frac{1}{i\hbar}[p_i, H] = \frac{1}{i\hbar}[p_i, V(\vec{x})]$$

But $[p_i, V(\vec{x})] = -i\hbar\frac{\partial V}{\partial x_i}$

i.e. $[\vec{p}, V(\vec{x})] = -i\hbar\vec{\nabla}V$

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i.e. $\boxed{\frac{d\vec{p}}{dt} = \frac{1}{i\hbar}\left(-i\hbar\vec{\nabla}V\right)} \boxed{= -\vec{\nabla}V} = \vec{F}$

Newton's 2nd Law

OR $\frac{d}{dt}\langle\vec{p}\rangle = -\langle\vec{\nabla}V\rangle$

"Ehrenfest Theorem"