Phys 5701 22 Oct 2020

Symmetry and Groups

S = Symmetry Operator Unitary

H is "symmetric" in S if $S^\dagger H S = H$

$${}_S\langle\alpha|H|\alpha\rangle_S = \langle\alpha|S^\dagger H S|\alpha\rangle = \langle\alpha|H|\alpha\rangle$$

Suppose $S = S(x)$ x = continuous parameter

$\Rightarrow S = 1 - \frac{i}{\hbar} G\, dx$

with $G^\dagger = G$ = "Generator"

$$S^\dagger H S = [1 + \tfrac{i}{\hbar} G\, dx]\, H\, [1 - \tfrac{i}{\hbar} G\, dx]$$

$$= H + \frac{i}{\hbar}(GH - HG)\, dx + O(dx^2) = H$$

$$\Rightarrow [G, H] = 0! \Rightarrow \frac{dG(t)}{dt} = 0$$

i.e. "G is conserved"

Orthogonal Group in 3D : SO(3)

Set of real 3x3 orthogonal matrices R

i.e. $R^T R = 1$

- R_1, R_2 then $R_1 R_2$ is an element

$$(R_1R_2)^T (R_1R_2) = R_2^T R_1^T R_1 R_2 = R_2^T R_2 = 1$$

- $R_1 (R_2 R_3) = (R_1 R_2) R_3$
- $\exists\ 1 : R1 = 1R = R$
- $R^{-1}\ \forall\ R^{-1}R = 1 \quad R^{-1} = R^T$

$\Rightarrow$ "Group O(3)" also $\det R = 1$ "S"

Does Nature realize SO(3)?

Yes!! Rotational invariance!

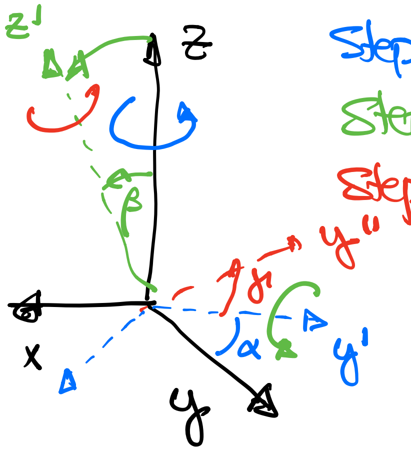
i.e. $D^{\dagger}(R)\, H\, D(R) = H$

$\Rightarrow H = \frac{1}{2m}\vec{P}^2 + V(r = |\vec{x}|)$

$\Rightarrow [\vec{J}, H] = 0$

i.e. Angular Momentum Conserved

Euler Rotations "Euler Angles"



Step#1: Rotate about z through α

Step#2: Rotate about y' through β

Step#3: Rotate about z' through γ

$$\Rightarrow R(\alpha, \beta, \gamma) = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha)$$

Good! Bad!! No generators for y', z'!!

But can do $R_{y'}(\beta)$ as follows:

- Undo z-rotation: $R_z^{-1}(\alpha)$
- Do β about y : $R_y(\beta)$
- Redo z rotation : $R_z(\alpha)$

$$\Rightarrow R_{y'}(\beta) = R_z(\alpha)\, R_y(\beta)\, R_z^{-1}(\alpha)$$

also $R_{z'}(\gamma) = R_{y'}(\beta)\, R_z(\gamma)\, R_{y'}^{-1}(\beta)$ ✓

$$\measuredangle D \quad R(\alpha,\beta,\gamma) = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha)$$

$$= R_{y'}(\beta) R_z(\gamma) \overbrace{R^{-1}_{y'}(\beta) R_{y'}(\beta)}^{1} R_z(\alpha)$$

$$= R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) R_z(\gamma) R_z(\alpha)$$

$$= R_z(\alpha) R_y(\beta) R_z(\gamma)$$

i.e. $D(\alpha,\beta,\gamma) = D_z(\alpha)\, \boxed{D_y(\beta)}\, D_z(\gamma)$

Example: Spin - $1/2$ (w/ Pauli Formalism)

$$\boxed{\exp\left[-\frac{i\phi}{2}\vec{\sigma}\cdot\hat{n}\right] = 1\cos(\phi/2) - i\,\vec{\sigma}\cdot\hat{n}\,\sin(\phi/2)}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$D(\alpha,\beta,\gamma) \doteq$$

$$= \begin{bmatrix} e^{i(\alpha+\gamma)/2}\cos\beta/2 & -e^{-i(\alpha-\gamma)/2}\sin\beta/2 \\ e^{i(\alpha-\gamma)/2}\sin\beta/2 & e^{i(\alpha+\gamma)/2}\cos\beta/2 \end{bmatrix}$$