Phys 5701 20 Oct 2020

From Last Class: Rotations

$|\alpha\rangle_R = \mathcal{D}(\hat{n},\phi)|\alpha\rangle = \mathcal{D}(R)|\alpha\rangle$

"Rotate state thru ϕ about axis $\hat{n}$"

$\phi, \hat{n}$ specify Rotation Matrix R

$$\mathcal{D}(\hat{n}, d\phi) = \underline{1} - \frac{i}{\hbar}\vec{J}\cdot\hat{n}\, d\phi$$

$$\vec{J} = J_x\hat{x} + J_y\hat{y} + J_z\hat{z}$$

Properties $R \Rightarrow [J_x, J_y] = i\hbar J_z$

also $[J_z, J_x] = i\hbar J_y \quad [J_y, J_z] = i\hbar J_x$

"Angular Momentum Commutation Relations"

"Argued" ${}_R\langle\alpha|J_x|\alpha\rangle_R = \langle\alpha|J_x|\alpha\rangle\cos\phi - \langle\alpha|J_y|\alpha\rangle\sin\phi$

For $R = R(\hat{z},\phi)$ "Right!"

In general $\langle J_k\rangle_R = \sum_{\ell=1,2,3} R_{k\ell}\langle J_\ell\rangle$

"Expectation value transforms like a vector."

Today: Spin - 1/2

$$S_x = \frac{\hbar}{2}\left\{ |+\rangle\langle -| + |-\rangle\langle +| \right\}$$

$$\boxed{S_y = \frac{i\hbar}{2}\left\{ -|+\rangle\langle -| + |-\rangle\langle +| \right\}}$$

$$S_z = \frac{\hbar}{2}\left\{ |+\rangle\langle +| - |-\rangle\langle -| \right\}$$

i.e. $S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$

These satisfy ang. mom. comm. relations!

$$|\alpha\rangle = \mathbb{1}|\alpha\rangle = |+\rangle\langle +|\alpha\rangle + |-\rangle\langle -|\alpha\rangle$$

$$|\alpha\rangle_R = \mathcal{D}_z(\phi)|\alpha\rangle = \mathcal{D}(\hat{z},\phi)|\alpha\rangle$$

$$= \exp\left[-\frac{i}{\hbar} S_z \phi\right] |\alpha\rangle$$

$$= e^{-iS_z\phi/\hbar} |+\rangle\langle +|\alpha\rangle + e^{-iS_z\phi/\hbar} |-\rangle\langle -|\alpha\rangle$$

$$= e^{-i\phi/2} |+\rangle\langle +|\alpha\rangle + e^{+i\phi/2} |-\rangle\langle -|\alpha\rangle$$

then $e^{\pm i\phi/2} = \cos\left(\frac{\phi}{2}\right) \pm i \sin\left(\frac{\phi}{2}\right)$

etc...

NOTE: For $\phi = 2\pi$ THEN $\exp\left(\pm i\frac{\phi}{2}\right) = -1$

$$\Rightarrow \mathcal{D}_z(2\pi)\,|\alpha\rangle = -|\alpha\rangle$$

"STRANGE BUT TRUE"

Spin Precession:

Electron in $\vec{B} = B\hat{z}$

$$H = -\underbrace{\left(\frac{e}{mc}\right)\vec{S}}_{\vec{\mu}}\cdot\vec{B} \equiv \omega S_z$$

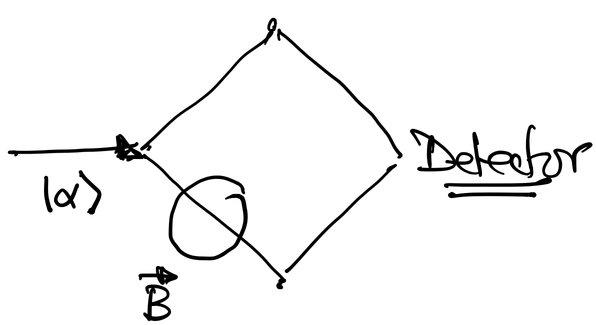
$$\omega \equiv |e|B/mc$$

$$\Rightarrow \mathcal{U}(t) = \exp\left[-\frac{i}{\hbar}Ht\right]$$

$$= \exp\left[-\frac{i}{\hbar}\omega t S_z\right]$$

$$= \mathcal{D}_z(\phi = \omega t)$$

$\Rightarrow$ Experiment! SEE POSTED PAPERS



Expect interference with period of 4π rotation.

Pauli Formalism : $|\pm\rangle$ representation

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_+ \Rightarrow \langle +| \doteq (1,0) \equiv \chi_+^\dagger$$

$$|-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_- \Rightarrow \langle -| \doteq (0,1) \equiv \chi_-^\dagger$$

$$|\alpha\rangle = |+\rangle\langle +|\alpha\rangle + |-\rangle\langle -|\alpha\rangle$$

$$\doteq \begin{pmatrix} \langle +|\alpha\rangle \\ \langle -|\alpha\rangle \end{pmatrix} \equiv \chi$$

"Spinors"

$$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \qquad c_\pm \equiv \langle \pm|\alpha\rangle$$

$$= c_+ \chi_+ + c_- \chi_-$$

$$\chi^\dagger = (c_+^*, c_-^*)$$

$$c_\pm^* = \langle \pm|\alpha\rangle^* = \langle \alpha|\pm\rangle$$

$$S_k \doteq \begin{bmatrix} \langle + | S_k | + \rangle & \langle + | S_k | - \rangle \\ \langle - | S_k | + \rangle & \langle - | S_k | - \rangle \end{bmatrix} \equiv \underbrace{\frac{\hbar}{2}\sigma_k}_{k=1,2,3}$$

$\sigma \equiv$ "Pauli Matrices"

$$\text{e.g. } S_y = \begin{bmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{bmatrix} = \frac{\hbar}{2} \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hermitian!

$$\sigma_i^2 = 1 \qquad [\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$$

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 0 \quad i \neq j$$

Ex $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

$$\det \sigma_i = -1 \qquad \mathrm{Tr}(\sigma_i) = 0$$

$$\vec{a} = a_1 \hat{x} + a_2 \hat{y} + a_3 \hat{z} \qquad a_i \ \underline{\underline{\text{Real}}}. \ (b_i, \text{too})$$

$$\vec{\sigma}\cdot\vec{a} = \begin{pmatrix} a_3 & a_1 - i a_2 \\ a_1 + i a_2 & -a_3 \end{pmatrix}$$

$$\Rightarrow (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\,(\mathbb{1}) + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$\Rightarrow \boxed{(\vec{\sigma}\cdot\vec{a})^2 = |\vec{a}|^2} \leftarrow$$

$$D(\hat{n},\phi) \overset{?}{\doteq} \exp\left[-\frac{i}{\hbar}\vec{S}\cdot\hat{n}\,\phi\right]$$

$$\overset{?}{\doteq} \exp\left[-i\,\vec{\sigma}\cdot\hat{n}\,\frac{\phi}{2}\right]$$

$$= \mathbb{1} - i(\vec{\sigma}\cdot\hat{n})\frac{\phi}{2} + \frac{(-i)^2}{2!}\left(\frac{\phi}{2}\right)^2(\vec{\sigma}\cdot\hat{n})^2 + \frac{(-i)^3}{3!}\left(\frac{\phi}{2}\right)^3(\vec{\sigma}\cdot\hat{n})^3 + \dots$$

$$(\vec{\sigma}\cdot\hat{n})^2 = |\hat{n}|^2 = 1$$

$$(\vec{\sigma}\cdot\hat{n})^{\text{even}} = 1$$

$$(\vec{\sigma}\cdot\hat{n})^{\text{odd}} = \vec{\sigma}\cdot\hat{n}$$

$\Rightarrow \exp\left[-i\,\vec{\sigma}\cdot\hat{n}\,\frac{\phi}{2}\right]$

$$= 1 - \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\phi}{2}\right)^4 + \cdots$$

$$-i(\vec{\sigma}\cdot\hat{n})\left[\frac{\phi}{2} - \frac{1}{3!}\left(\frac{\phi}{2}\right)^3 + \cdots\right]$$

$$= \cos(\phi/2) - i(\vec{\sigma}\cdot\hat{n})\sin(\phi/2)$$

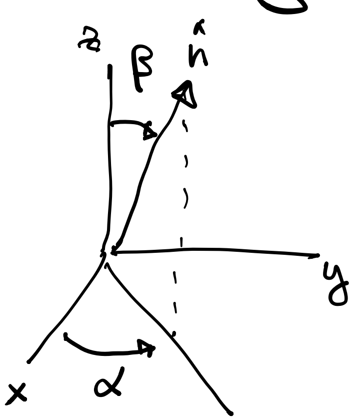
Rotation operator represented in $|\pm\rangle$ basis!

$$D(\hat{n},\phi)|\alpha\rangle \doteq \left[1\cos\frac{\phi}{2} - i(\vec{\sigma}\cdot\hat{n})\sin\frac{\phi}{2}\right]\chi$$

Recall Prob 1.11 (HW #2 Prob. 1)

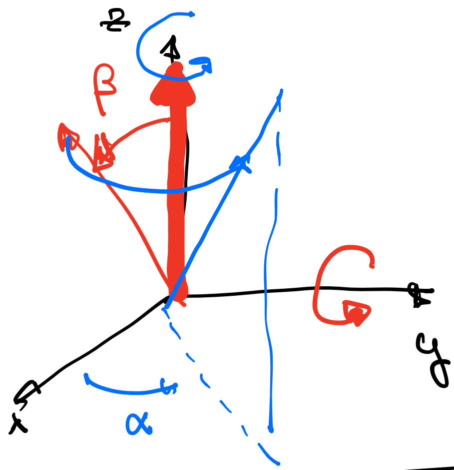
Eigenvalue problem

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = +\frac{\hbar}{2}\,|\vec{S}\cdot\hat{n};+\rangle$$



$$\boxed{|\vec{S}\cdot\hat{n};+\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + e^{i\alpha}\sin\left(\frac{\beta}{2}\right)|-\rangle}$$

Now: Do it with Rotations!!



1) Rotate thru β about y

2) Rotate thru α about z

$$\underline{D_z(\alpha)}\, D_y(\beta)\,(|\alpha\rangle = |+\rangle)$$

$$\doteq \left[1\cos\frac{\alpha}{2} - i\sigma_z \sin\frac{\alpha}{2}\right]\left[1\cos\frac{\beta}{2} - i\sigma_y \sin\frac{\beta}{2}\right]\chi_+$$

$$= \begin{bmatrix} \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2} \end{bmatrix}\begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{+i\alpha/2} \end{bmatrix}\begin{bmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{bmatrix} = \begin{bmatrix} e^{-i\alpha/2}\cos\frac{\beta}{2} \\ e^{+i\alpha/2}\sin\frac{\beta}{2} \end{bmatrix}$$

$$\doteq e^{-i\alpha/2}\cos\frac{\beta}{2}\,|+\rangle + e^{+i\alpha/2}\sin\frac{\beta}{2}\,|-\rangle$$

$$= e^{-i\alpha/2} \times (\text{Solution to Prob. 1.11})$$