Phys 5701 19 Nov 2020

Two Independent Harmonic Oscillators

$[a_+, a_+^\dagger] = 1$ w/ $[a_+, a_-] = 0$

$[a_-, a_-^\dagger] = 1$

$N_+ = a_+^\dagger a_+$ $N_- = a_-^\dagger a_-$

w/ $[N_+, N_-] = 0$

$\Rightarrow$ $N_+ |n_+, n_-\rangle = n_+ |n_+, n_-\rangle$

$N_- |n_+, n_-\rangle = n_- |n_+, n_-\rangle$

But $[N_+, a_+] = -a_+$ etc...

$\Rightarrow$ $n_+ = 0, 1, 2, \ldots$
$n_- = 0, 1, 2 \ldots$

$|0,0\rangle$ = "Vacuum State" $\equiv |\mathbf{0}\rangle$

Recall $a_+^\dagger |n_+, n_-\rangle = \sqrt{n_+ + 1}\, |n_+ + 1, n_-\rangle$ $+1$

$a_+ |n_+, n_-\rangle = \sqrt{n_+}\, |n_+ - 1, n_-\rangle$ -1

$$a_+ |0,0\rangle = 0 = a_- |0,0\rangle$$

$$|n_+, n_-\rangle = \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}} \frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}} |0,0\rangle$$

Associate with Angular Momentum

Define $J_z \equiv \frac{\hbar}{2}[N_+ - N_-]$ $\leftarrow$

$$J_z |n_+, n_-\rangle = \frac{\hbar}{2}(n_+ - n_-)|n_+, n_-\rangle$$

$\rightarrow m\hbar$ i.e. $n_+ - n_- \rightarrow 2m$

Can show that for $J_+ \equiv \hbar a_+^\dagger a_-$

$J_- \equiv \hbar a_-^\dagger a_+$

have $\boxed{[J_+, J_-] = 2\hbar J_z}$

& $[J_z, J_\pm] = \pm\hbar J_\pm$

} Equivalent to $[J_x, J_y] = i\hbar J_z$ etc...

Proof: $[J_+, J_-] = \hbar^2 [a_+^\dagger a_-, a_-^\dagger a_+]$

$$= \hbar^2 (a_+^\dagger a_- a_-^\dagger a_+ - a_-^\dagger a_+ a_+^\dagger a_-)$$

But $a_- a_-^\dagger - a_-^\dagger a_- = 1 \Rightarrow a_- a_-^\dagger = 1 + a_-^\dagger a_-$

$$\Rightarrow [J_+, J_-] = \hbar^2 (a_+^\dagger a_+ + a_+^\dagger a_-^\dagger a_- a_+ - a_-^\dagger a_- - a_-^\dagger a_+^\dagger a_+ a_-)$$

$$= \hbar^2 (a_+^\dagger a_+ - a_-^\dagger a_-)$$

$$= 2\hbar J_z$$ QED.

Define $N = N_+ + N_-$

$$N |n_+, n_-\rangle = (n_+ + n_-) |n_+, n_-\rangle$$

$\Rightarrow$ Try $\vec{J}^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2$

$$= \frac{1}{2}\hbar^2 (a_+^\dagger a_- a_-^\dagger a_+ + a_-^\dagger a_+ a_+^\dagger a_-) + \frac{1}{4}\hbar^2 (N_+ - N_-)^2$$

$$a_+^\dagger a_- a_-^\dagger a_+ + a_-^\dagger a_+ a_+^\dagger a_-$$

$$= a_+^\dagger a_+ + a_+^\dagger a_-^\dagger a_- a_+$$

$$+ a_-^\dagger a_- + a_-^\dagger a_+^\dagger a_+ a_-$$

$$= N_+ + N_+ N_-$$

$$\Rightarrow + N_- + N_- N_+ = N_+ + N_- + 2 N_+ N_-$$

$$\vec{J}^2 = \frac{\hbar^2}{4}\left[2N_+ + 2N_- + \cancel{4N_+N_-} + N_+^2 \cancel{+} 2N_+N_- + N_-^2\right]$$

$$= \frac{\hbar^2}{4}\left[2N + N^2\right] = \frac{\hbar^2}{2} N \left(\frac{N}{2}+1\right)$$

"Spin zero has $0 = n_+ + n_-$ quanta"

$\leadsto$ $\vec{J}^2$ eigenvalue $= 0$

"Spin-1/2 has $1 = n_+ + n_-$ quanta"

$\leadsto$ $\vec{J}^2$ eigenvalue $\frac{3}{4}\hbar^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2$

"Spin-1" $2 = n_+ + n_-$

$\leadsto$ $\vec{J}^2$ eigenvalue $\frac{\hbar^2}{2} 2(1+1) = 1(1+1)\hbar^2$.

More formally

$$m \to (n_+ - n_-)/2$$

$$j \to (n_+ + n_-)/2$$

$\Rightarrow$ $n_+ \to j+m$ $\quad n_- \to j-m$

i.e. Eigenvalues of $\vec{J}^2$ are

$$\frac{\hbar^2}{2}\, 2j\left(\frac{2j}{2}+1\right) = j(j+1)\hbar^2$$ Right!!

$$|n_+, n_-\rangle = \frac{(a_+^+)^{n_+}}{\sqrt{n_+!}} \frac{(a_-^+)^{n_-}}{\sqrt{n_-!}} |0,0\rangle$$

$$|j\,m\rangle = \frac{(a_+^+)^{j+m} (a_-^+)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} |0,0\rangle$$

$$\Rightarrow |j\,j\rangle = \frac{(a_+^+)^{2j}}{\sqrt{(2j)!}} |0\rangle$$

Useful! Calculating $D^{(j)}_{m'm}(R)$

Recall $D^{(j)}_{m'm}(\alpha,\beta,\gamma)$

$$\equiv \langle jm' | D(R) | jm\rangle$$

$$D(R)|jm\rangle = D(R)\frac{(a_+^\dagger)^{j+m}}{\sqrt{(j+m)!}}\frac{(a_-^\dagger)^{j-m}}{\sqrt{(j-m)!}}\;\overset{\uparrow}{4}\;|0,0\rangle$$

$$a_+^\dagger a_+^\dagger \cdots a_+^\dagger{\downarrow} a_-^\dagger{\downarrow} a_-^\dagger{\downarrow} \cdots {\downarrow} a_-^\dagger{\downarrow} \qquad 1 = D^{-1}(R)\,D(R)$$

$$\Rightarrow D(R)|jm\rangle = \frac{[D(R)a_+^\dagger D^{-1}(R)]^{j+m}\,[D(R)a_-^\dagger D^{-1}(R)]^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}$$

$$D(R)\,|0,0\rangle$$

$D(R)\,a_\pm^\dagger\,D^{-1}(R)$

use $D(0,\beta,0) = e^{-iJ_y\beta/\hbar}$ $\qquad J_y = \frac{1}{2i}(J_+ - J_-)$

then Baker-Hausdorf lemma... $\Rightarrow$ (3.426)