PHYS 5701 17 SEP 2020

Today: Quantum Dynamics

aka "Time Evolution"

Time is a parameter that marks change.

Time is not a quantum mechanical observable

$|\alpha, t_0; t\rangle \quad (t > t_0)$ "state at time t"

$|\alpha, t_0; t_0\rangle = |\alpha\rangle \equiv |\alpha, t_0\rangle$

Time Evolution Operator $\mathcal{U}(t, t_0)$

i.e. $|\alpha, t_0; t\rangle = \mathcal{U}(t, t_0)\,|\alpha, t_0\rangle$

$$\langle \alpha, t_0; t | \alpha, t_0; t\rangle = \langle \alpha, t_0 | \alpha, t_0\rangle$$

$$= \left[\langle \alpha, t_0| \mathcal{U}^\dagger\right]\left[\mathcal{U}|\alpha, t_0\rangle\right]$$

$$= \langle \alpha, t_0| \mathcal{U}^\dagger \mathcal{U} |\alpha, t_0\rangle$$

$\Rightarrow \mathcal{U}^\dagger(t, t_0)\,\mathcal{U}(t, t_0) = 1$ UNITARY!

Observable A w/ $A|a'\rangle = a'|a'\rangle$

$$|\alpha, t_0\rangle = \sum_{a'} c_{a'}(t_0)\, |a'\rangle$$

$$|\alpha, t_0; t\rangle = \sum_{a'} c_{a'}(t)\, |a'\rangle$$

$$|c_{a'}(t)| \neq |c_{a'}(t_0)| \text{ in general}$$

But $$\sum_{a'} |c_{a'}(t)|^2 = \sum_{a'} |c_{a'}(t_0)|^2$$

<u>Constructing $\mathcal{U}(t, t_0)$</u>

For $t = t_0 + dt$

$$\Rightarrow \mathcal{U}(t_0 + dt, t_0) = 1 - i\Omega\, dt$$

$$\text{w/ } \Omega^+ = \Omega$$

$\Omega = \frac{H}{\hbar}$ i.e. $$\boxed{\mathcal{U}(t_0 + dt, t_0) = 1 - \frac{i}{\hbar} H\, dt}$$

$H =$ <u>Hamiltonian</u> Eigenvalues = "Energies"

$$\mathcal{U}(t_2, t_0) = \mathcal{U}(t_2, t_1)\,\mathcal{U}(t_1, t_0)$$

$$\mathcal{U}(t+dt, t_0) = \mathcal{U}(t+dt, t)\,\mathcal{U}(t, t_0)$$

$$= \left[1 - \frac{i}{\hbar} H\,dt\right] \mathcal{U}(t, t_0)$$

$$\frac{\mathcal{U}(t+dt, t_0) - \mathcal{U}(t, t_0)}{dt} = -\frac{i}{\hbar} H\,\mathcal{U}(t, t_0)$$

$$\boxed{i\hbar \frac{\partial}{\partial t} \mathcal{U}(t, t_0) = H\,\mathcal{U}(t, t_0)}$$

"Schrödinger Equation"

$\times\ |\alpha, t_0\rangle$ then...

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0; t\rangle = H\,|\alpha, t_0; t\rangle$$

<u>then</u>... project onto...

$\langle \pm|$

<u>OR</u> $\langle \vec{x}'|$

OR ...

Solving for $\mathcal{U}(t,t_0)$

1) $H(t) = H(t_0) = H$

$\Rightarrow$ $$\boxed{\mathcal{U}(t,t_0) = \exp\left[-\frac{i}{\hbar} H (t-t_0)\right]}$$

2) $[H(t_1), H(t_2)] = 0$

3) $H(t)$ DO NOT COMMUTE.

FREEMAN DYSON

Phys. Rev. **75** (1949) 486

Energy Eigenstates

Observable A w/ $A|a'\rangle = a'|a'\rangle$

also $[A,H]=0$

$\Rightarrow$ $H|a'\rangle = E_{a'}|a'\rangle$

$$\mathcal{U}(t,t_0) = \left[\sum_{a''} |a''\rangle\langle a''|\right]$$

$$\left[\exp\left(-\frac{i}{\hbar} H (t-t_0)\right)\right]\left[\sum_{a'} |a'\rangle\langle a'|\right]$$

But $e^{-iH(t-t_0)/\hbar}|a'\rangle = e^{-iE_{a'}(t-t_0)/\hbar}|a'\rangle$

and $\langle a''|a'\rangle = \delta_{a',a''}$

$\nexists_D\ \mathcal{U}(t,t_0) = \sum_{a'} |a'\rangle e^{-iE_{a'}(t-t_0)/\hbar}\langle a'|$

$$\boxed{\begin{gathered} |\alpha, t_0=0; t\rangle = \sum_{a'} |a'\rangle e^{-iE_{a'}t/\hbar}\langle a'|\alpha\rangle \\ C_{a'}(t) = e^{-iE_{a'}t/\hbar}\, C_{a'}(\cancel{0}) \end{gathered}}$$

<u>Expectation Values</u> $[A,H]=0$

But $[A,B]\neq 0$

(1) For $|\alpha, t_0=0\rangle = |a'\rangle$

$\langle\alpha, t_0=0; t|B|\alpha, t_0=0.t\rangle$

$= \left[\langle a'|e^{+iE_{a'}t/\hbar}\right] B \left[e^{-iE_{a'}t/\hbar}|a'\rangle\right]$

$= \langle a'|B|a'\rangle$ NO CHANGE WITH TIME

(2) For arbitrary initial state

$$|\alpha, t_0=0; t\rangle = \sum_{a'} c_{a'}\, e^{-iE_{a'}t/\hbar}\, |a'\rangle$$

$$\langle B\rangle = \langle \alpha, t_0=0; t|B|\alpha, t_0=0; t\rangle$$

$$= \sum_{a'} \sum_{a''} c_{a'}^{*} c_{a''}\, e^{-i(E_{a''}-E_{a'})t/\hbar} \langle a'|B|a''\rangle$$

Terms "interfere" w/ freq $\omega = \frac{\Delta E}{\hbar}$

<u>Physics Example: Spin Precession</u>

Energy of $\vec{\mu}$ in $\vec{B}$ $= -\vec{\mu}\cdot\vec{B}$

For an electron $\vec{\mu} = e\vec{S}/mc$ $(e<0)$

$$\Rightarrow H = -\frac{e}{mc}\vec{S}\cdot\vec{B} = -\frac{eB}{mc}S_z$$

"A" $= S_z$ $\quad \{|a'\rangle\} = |\pm\rangle$

$$\Rightarrow E_{a'} = E_{\pm} = \mp\frac{e\hbar B}{2mc}$$

$$\omega \equiv \frac{|e|B}{mc}$$ "Larmor Frequency"

$\Rightarrow H = \omega S_z \qquad E_\pm = \pm \hbar\omega/2$

$$\mathcal{U}(t,t_0) = \exp[-i\omega t S_z/\hbar]$$

w/ $S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$

$|\alpha, t_0 = 0\rangle = |\alpha\rangle = c_+|+\rangle + c_-|-\rangle$

$\Rightarrow |\alpha, t_0=0; t\rangle = c_+ e^{-i\omega t/2} |+\rangle + c_- e^{+i\omega t/2} |-\rangle$

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(1) $|\alpha, t_0 = 0; t\rangle = |+\rangle$ "spin up"

i.e. $c_+ = 1 \qquad c_- = 0$

$|\alpha, t_0 = 0; t\rangle = e^{-i\omega t/2} |+\rangle$

Stays in same state!

Phase factor included.

(2) $|\alpha, t_0=0; t\rangle = |S_x; +\rangle$

i.e. $c_+ = \frac{1}{\sqrt{2}} = c_-$

$$|\alpha, t_0=0; t\rangle = \frac{1}{\sqrt{2}} e^{-i\omega t/2}|+\rangle + \frac{1}{\sqrt{2}} e^{+i\omega t/2}|-\rangle$$

- What is the probability <u>as a function of time</u> that I measure the state $S_x; +$?

$$|\langle S_x; + | \alpha, t_0=0; t\rangle|^2$$

$$= \left| \left[\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right]\left[\frac{1}{\sqrt{2}} e^{-i\omega t/2}|+\rangle + \frac{1}{\sqrt{2}} e^{i\omega t/2}|-\rangle\right]\right|^2$$

$$= \frac{1}{4}\left| [\langle +| + \langle -|]\,[e^{-i\omega t/2}|+\rangle + e^{+i\omega t/2}|-\rangle]\right|^2$$

$$= \frac{1}{4}\left| e^{-i\omega t/2} + 0 + 0 + e^{+i\omega t/2}\right|^2$$

$$= \frac{1}{4}\left|2\cos\frac{\omega t}{2}\right|^2$$

$$= \cos^2\left(\frac{\omega t}{2}\right) = \frac{1}{2}\left[1 + \cos(\omega t)\right]$$

period $2\pi/\omega$

• What is $\langle S_x \rangle$

$$\langle \alpha, t_0=0; t | \underline{\underline{S_x}} | \alpha, t_0=0; t \rangle$$

sup

$$= \left[\frac{1}{\sqrt{2}} e^{+i\omega t/2} \langle +| + \frac{1}{\sqrt{2}} e^{-i\omega t/2} \langle -| \right]$$

$$\left[\frac{\hbar}{2} \left(-i|+\rangle\langle -| + i|-\rangle\langle +| \right) \right]$$

$$\left[\frac{1}{\sqrt{2}} e^{-i\omega t/2} |+\rangle + \frac{1}{\sqrt{2}} e^{+i\omega t/2} |-\rangle \right]$$

$$= \frac{\hbar}{4} \left[e^{+i\omega t/2} \langle -| + e^{-i\omega t/2} \langle +| \right]$$

$$\times \left[e^{-i\omega t/2} |+\rangle + e^{i\omega t/2} |-\rangle \right]$$

$$= \frac{\hbar}{4} \left[0 + e^{i\omega t} + e^{-i\omega t} + 0 \right]$$

$$= \frac{\hbar}{2} \cos(\omega t)$$

$\sin(\omega t)$

"Right!"