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Angular Momentum Addition

The Goal Today: Two $\vec{J}$'s $\vec{J}_1$, $\vec{J}_2$

Different spaces!

i.e. $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$

Examples
- $\vec{S}_p$, $\vec{S}_e$
- $\vec{L}$, $\vec{S}$ for an electron in atom

$\Rightarrow$ Want eigenvalues and eigenvectors for $\vec{J} = \vec{J}_1 + \vec{J}_2$

$$\vec{J} = \vec{J}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \vec{J}_2$$

Nobody ever writes it this way!

$\Leftarrow$ Finding eivs of $\vec{J}^2$ and J_z in the basis

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle \rightarrow |j_1 j_2 ; m_1 m_2\rangle$$

Formalism of $\vec{J} = \vec{J}_1 + \vec{J}_2$

Have: $[J_{1_i}, J_{1_j}] = i\hbar\, \varepsilon_{ijk} J_{1_k}$

$[J_{2_i}, J_{2_j}] = i\hbar\, \varepsilon_{ijk} J_{2_k}$

$[J_{1_i}, J_{2_j}] = 0$

Now $[J_i, J_j] = [J_{1_i} + J_{2_i}, J_{1_j} + J_{2_j}]$

$= [J_{1_i}, J_{1_j}] + [J_{2_i}, J_{2_j}] + 0 + 0$

$= i\hbar\, \varepsilon_{ijk} J_{1_k} + i\hbar\, \varepsilon_{ijk} J_{2_k}$

$= i\hbar\, \varepsilon_{ijk} (J_{1_k} + J_{2_k}) = i\hbar\, \varepsilon_{ijk} J_k$

$\Rightarrow \vec{J}$ is also angular momentum!

Next: $[\vec{J}^2, \vec{J}_1^2] = [\cancel{\vec{J}_1^2, \vec{J}_1^2}]^{=0} + [\cancel{\vec{J}_2^2, \vec{J}_1^2}]^{=0}$

$+ [2\vec{J}_1 \cdot \vec{J}_2, \vec{J}_1^2] \;\; " = 0$

$= 0 = [\vec{J}^2, \vec{J}_2^2]$

and $[J_2, \vec{J}_1^2] = [J_{1_2}, \vec{J}_1^2] + [J_{2_2}, \vec{J}_1^2]$

$= 0 = [J_2, \vec{J}_2^2]$

However $[\vec{J}^2, J_{1z}] = [2\underbrace{\vec{J}_1}_{\text{contains } J_x, J_y} \cdot \vec{J}_2, J_{1z}] \neq 0$

so (A)

$$\vec{J}_1^2 |j_1 j_2; m_1 m_2\rangle = j_1(j_1+1)\hbar^2 |j_1 j_2; m_1 m_2\rangle$$

$$J_{1z} |j_1 j_2; m_1 m_2\rangle = m_1\hbar \, |j_1 j_2; m_1 m_2\rangle$$

$$\vec{J}_2^2 |j_1 j_2; m_1 m_2\rangle = j_2(j_2+1)\hbar^2 |j_1 j_2; m_1 m_2\rangle$$

$$J_{2z} |j_1 j_2; m_1 m_2\rangle = m_2\hbar \, |j_1 j_2; m_1 m_2\rangle$$

OR (B)

$$\vec{J}_1^2 |j_1 j_2; j m\rangle = j_1(j_1+1)\hbar^2 \, |j_1 j_2; j m\rangle$$

$$\vec{J}_2^2 |j_1 j_2; j m\rangle = j_2(j_2+1)\hbar^2 \, |j_1 j_2; j m\rangle$$

$$\vec{J}^2 |j_1 j_2; j m\rangle = j(j+1)\hbar^2 \, |j_1 j_2; j m\rangle$$

$$J_z |j_1 j_2; j m\rangle = m\hbar \, |j_1 j_2; j m\rangle$$

i.e. (A) = Basis $|j_1 j_2; m_1 m_2\rangle$

(B) = Basis $|j_1 j_2; j m\rangle$

w/ $\mathbb{1} = \sum_{m_1} \sum_{m_2} |j_1 j_2; m_1 m_2\rangle \langle j_1 j_2; m_1 m_2|$

so $|j_1 j_2; j m\rangle$ $\leftarrow$

$$= \sum_{m_1} \sum_{m_2} |j_1 j_2; m_1 m_2\rangle\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle$$

"Clebsch-Gordan Coeff's"

Simple Example #1: Two Spin-1/2

$$|j_1 j_2; m_1 m_2\rangle = \{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$$

Find 4x4 matrix representation of

$$\vec{S}^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\vec{S}_1 \cdot \vec{S}_2$$

$$= 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

$\leadsto$ Diagonalize!

Simple Example #2: $\vec{L}$, $\vec{S}$

$$|\vec{x}'; \pm\rangle = |\vec{x}'\rangle \otimes |\pm\rangle$$

$$\psi(\vec{x}) = \begin{bmatrix} \psi_+(\vec{x}) \\ \psi_-(\vec{x}) \end{bmatrix}$$

Orbital $\vec{L}$ eigenstates

$|l m_l \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$

Properties of Clebsch-Gordan Coefficients

$$J_z = J_{1z} + J_{2z}$$

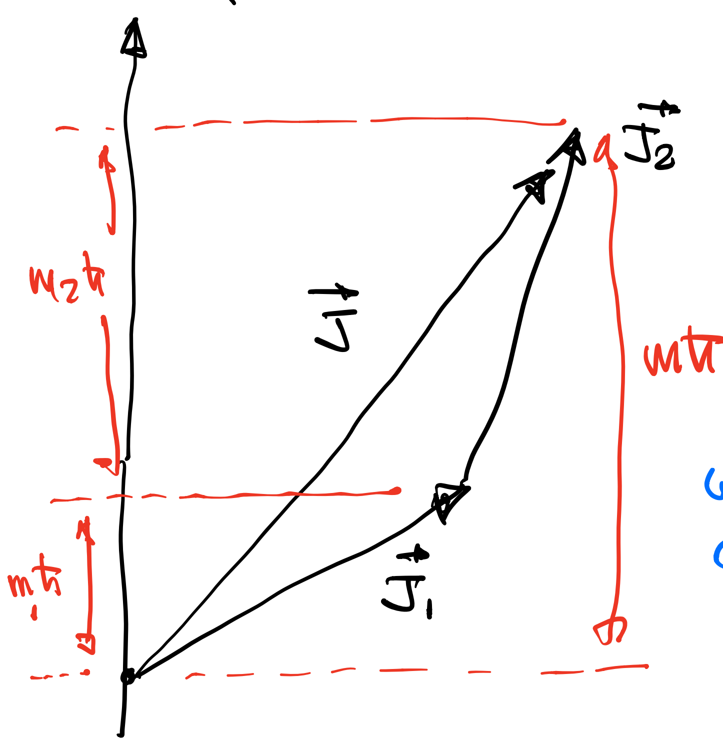
$$\Leftrightarrow \quad [J_z - J_{1z} - J_{2z}]\,|j_1 j_2; jm\rangle = 0$$

$$\langle j_1 j_2; m_1 m_2|\,[J_z - J_{1z} - J_{2z}]\,|j_1 j_2; jm\rangle = 0$$

$$(m - m_1 - m_2)\,\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle = 0$$

i.e. $m = m_1 + m_2$ OR $\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle = 0$

No surprise: think classically:



$\Rightarrow$ $\underline{\underline{m_1 + m_2 = m}}$.

what about constraints on magnitudes?

Claim: $\boxed{|j_1-j_2| \le j \le j_1+j_2}$ //

Test: Count # of states in each representation.

(A) $|j_1 j_2 ; m_1 m_2\rangle$: $N = (2j_1+1)(2j_2+1)$

(B) $|j_1 j_2 ; j m\rangle$ $N = \sum_{j=j_1-j_2}^{j_1+j_2} (2j+1)$

$$= 2\underbrace{\sum_{j=j_1-j_2}^{j_1+j_2} j} + \underbrace{(j_1+j_2)-(j_1-j_2)+1}_{2j_2+1}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\left[\begin{array}{l} 1 + 2 + \cdots + n \\ n + n-1 + \cdots + 1 \\ \quad =(n+1)n \end{array}\right]$$

$$\sum_{i=k}^{\ell} = \sum_{i=1}^{\ell} - \sum_{i=1}^{k-1} = \frac{\ell(\ell+1)}{2} - \frac{(k-1)k}{2}$$

$$2\sum_{j=j_1-j_2}^{j_1+j_2} j = (j_1+j_2)(j_1+j_2+1) - (j_1-j_2-1)(j_1-j_2)$$

$$= 2j_1 + 4j_1 j_2$$

$\Leftrightarrow N_B = 2j_1+4j_1j_2+2j_2+1 = (2j_1+1)(2j_2+1)$

$= N_A$ RIGHT!

Recursion Relations

$$J_{\pm} |jm\rangle = [(j \mp m)(j \pm m + 1)]^{1/2} |j, m \pm 1\rangle$$

$$|j_1 j_2 ; j = j_1 + j_2, m = \underline{\underline{j}}\rangle$$

$$= |j_1 j_2 ; m_1 = \underline{\underline{j_1}}, m_2 = \underline{\underline{j_2}}\rangle$$

then do $J_- = J_{1-} - J_{2-}$

SEE PROBLEMS: 3.34, 3.35

Useful Physics Power

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \Rightarrow \vec{J}^2 = \vec{J}_1^2 + \vec{J}_2^2 + 2\vec{J}_1 \cdot \vec{J}_2$$

$$\Rightarrow \underline{\underline{\vec{J}_1 \cdot \vec{J}_2}} = \frac{1}{2}\left[\underline{\underline{\vec{J}^2}} - \underline{\underline{\vec{J}_1^2}} - \underline{\underline{\vec{J}_2^2}}\right]$$

e.g. $\vec{S}_p \cdot \vec{S}_e$ "Hyperfine Interaction"

$\vec{L} \cdot \vec{S}$ "Spin-Orbit Interaction"

$\Rightarrow$ Evaluate Matrix Elems in $|j_1 j_2 ; jm\rangle$ basis!