Phys 5701   15 Sep 2020

From Last Class: Position & Momentum

$$X|x'\rangle = x'|x'\rangle \qquad \langle x'|x''\rangle = \delta(x'-x'')$$

$$1 = \int dx' \, |x'\rangle\langle x'|$$

Translations: $\mathcal{J}(dx')|x'\rangle = |x'+dx'\rangle$

$$\mathcal{J}(dx') = 1 - i p \, dx'/\hbar$$

$$p \equiv \text{"momentum"}$$

FINITE TRANSLATION

$$\mathcal{J}(x'+dx') = \mathcal{J}(x')\mathcal{J}(dx')$$
$$= \mathcal{J}(x')(1 - ip\,dx'/\hbar)$$

$$\Rightarrow \frac{\mathcal{J}(x'+dx') - \mathcal{J}(x')}{dx'} = -ip/\hbar$$

$$\frac{\partial \mathcal{J}(x')}{\partial x'} = -i\frac{p}{\hbar} \Rightarrow \mathcal{J}(x') = e^{-ipx'/\hbar}$$

Today: Wave Functions

$$\langle x' | \alpha \rangle = \psi_\alpha(x')$$

$$1 = \langle \alpha | \alpha \rangle = \int \langle \alpha | x' \rangle \langle x' | \alpha \rangle \, dx'$$

$$= \int \psi_\alpha^*(x') \psi_\alpha(x') \, dx'$$

Recall $\langle \beta | \alpha \rangle$

= "Probability amplitude for measuring the state $|\alpha\rangle$ to appear as the state $|\beta\rangle$"

$$= \int dx' \langle \beta | x' \rangle \langle x' | \alpha \rangle$$

$$= \int dx' \psi_\beta^*(x') \psi_\alpha(x')$$ "overlap integral"

Consider observable A

with $A|a'\rangle = a'|a'\rangle$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$$

$$\Leftrightarrow \psi_\alpha(x') = \sum_{a'} c_{a'} u_{a'}(x')$$

"Eigenfunction expansion"

$$u_{a'}(x') \equiv \langle x'|a'\rangle$$

Evaluating Matrix Elements

Consider operator $f(x) \equiv x^2$

$$\langle\beta|f(x)|\alpha\rangle$$

$$= \int dx' \int dx'' \langle\beta|x'\rangle\langle x'|x^2|x''\rangle\langle x''|\alpha\rangle$$

$$= \int dx' \int dx'' \langle\beta|x'\rangle x''^2 \underbrace{\langle x'|x''\rangle}_{} \langle x''|\alpha\rangle$$

$$= \delta(x'-x'')$$

But $\int_{-\infty}^{\infty} g(x)\,\delta(x-a)\,dx = g(a)$

$$\Rightarrow \langle \beta | f(x) | \alpha \rangle = \int dx' \, \psi_\beta^*(x') \, x'^2 \, \psi_\alpha(x')$$

For $f(x) = x^2$

But Taylor expand "anything"

$$\Rightarrow \boxed{\langle \beta | f(x) | \alpha \rangle = \int dx' \, \psi_\beta^*(x') \, f(x') \, \psi_\alpha(x')}$$

"Momentum Operator in Position Basis"

Better: What is $p|\alpha\rangle$ in x-basis?

$$\mathcal{J}(\Delta x') = 1 - ip\Delta x'/\hbar$$

$$(1 - ip\Delta x'/\hbar)|\alpha\rangle = \mathcal{J}(\Delta x')|\alpha\rangle$$

$$= \int dx' \, \mathcal{J}(\Delta x') |x'\rangle \langle x' | \alpha \rangle$$

$$= \int_{-\infty}^{\infty} dx' \, |x' + \Delta x'\rangle \langle x' | \alpha \rangle$$

$$= \int dx' \, |x'\rangle \langle x' - \Delta x' | \alpha \rangle$$

$$\langle x' - \Delta x' | \alpha \rangle = \langle x' | \alpha \rangle - \Delta x' \frac{\partial}{\partial x'} \langle x' | \alpha \rangle + \cdots$$

$$\left(1 - i p \Delta x' / \hbar\right) | \alpha \rangle$$

$$= \int dx' \, | x' \rangle \langle x' | \alpha \rangle$$

$$- \Delta x' \int dx' \, | x' \rangle \frac{\partial}{\partial x'} \langle x' | \alpha \rangle$$

$$- i \frac{p}{\hbar} | \alpha \rangle = - \int dx' \, | x' \rangle \frac{\partial}{\partial x'} \langle x' | \alpha \rangle$$

$$\frac{i}{\hbar} \langle x'' | p | \alpha \rangle = \int dx' \, \langle x'' | x' \rangle \frac{\partial}{\partial x'} \langle x' | \alpha \rangle$$

$$\Leftrightarrow \boxed{\langle x' | p | \alpha \rangle = \frac{\hbar}{i} \frac{\partial}{\partial x'} \langle x' | \alpha \rangle}$$

" $p \longrightarrow \frac{\hbar}{i} \frac{d}{dx}$ "

$$\langle\beta|p|\alpha\rangle = \int dx' \langle\beta|x'\rangle\langle x'|p|\alpha\rangle$$

$$= \int dx' \, \psi_\beta^*(x') \frac{\hbar}{i}\frac{\partial}{\partial x'} \psi_\alpha(x')$$

$$\langle\beta|p^n|\alpha\rangle = \int dx' \, \psi_\beta^*(x') \left(\frac{\hbar}{i}\right)^n \frac{\partial^n}{\partial x'^n} \psi_\alpha(x')$$

Momentum Space

$P|p'\rangle = p'|p'\rangle \qquad \langle p'|p''\rangle = \delta(p'-p'')$

Momentum space wave functions

$$\phi_\alpha(p') = \langle p'|\alpha\rangle$$

Good Question: What is $\langle x'|p'\rangle$?

$$\langle x'|p|p'\rangle = p'\langle x'|p'\rangle$$

also

$$= \frac{\hbar}{i}\frac{d}{dx'}\langle x'|p'\rangle$$

$$\langle x'|p'\rangle = e^{ip'x'/\hbar} \times \underline{\underline{N}}.$$

NORMALIZATION

$$\delta(x'-x'') = \langle x'|x''\rangle = \int dp'\,\langle x'|p'\rangle\langle p'|x''\rangle$$

$$= \int dp'\left(N e^{ip'x'/\hbar}\right)\left(N^* e^{-ip'x''/\hbar}\right)$$

$$= |N|^2 \int dp'\, e^{ip'(x'-x'')/\hbar}$$

Recall $\delta(x) = \frac{1}{2\pi} \int e^{ikx} dk$

so $1 = |N|^2 \, 2\pi\hbar$

so $$\boxed{\langle x' | p' \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ix'p'/\hbar}}$$

Position and Momentum Space

$$\langle x' | \alpha \rangle = \int dp' \langle x' | p' \rangle \langle p' | \alpha \rangle$$

i.e. $$\psi_\alpha(x') = \int dp' \frac{1}{\sqrt{2\pi\hbar}} e^{ix'p'/\hbar} \phi_\alpha(p')$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dp' \, e^{ix'p'/\hbar} \phi_\alpha(p')$$

FOURIER TRANSFORM $\left(k \equiv \frac{p'}{\hbar}\right)$

$$\langle p'|\alpha\rangle = \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$$

$$\phi_\alpha(p') = \frac{1}{\sqrt{2\pi\hbar}} \int dx' \, e^{-ix'p'/\hbar} \psi_\alpha(x')$$

INVERSE F.T.

Generalize to 3D

$$\langle \vec{x}'|\vec{x}''\rangle = \delta^{(3)}(\vec{x}' - \vec{x}'') \qquad |\vec{x}'\rangle = |x', y', z'\rangle$$

$$\langle \vec{p}'|\vec{p}''\rangle = \delta^{(3)}(\vec{p}' - \vec{p}'') \qquad |\vec{p}'\rangle = |p_x', p_y', p_z'\rangle$$

w/ $\vec{X}|\vec{x}'\rangle = \vec{x}'|\vec{x}'\rangle$ & $\vec{P}|\vec{p}'\rangle = \vec{p}'|\vec{p}'\rangle$

i.e. $x|\vec{x}'\rangle = x'|\vec{x}'\rangle$ i.e. $p_x|\vec{p}'\rangle = p_x'|\vec{p}'\rangle$

$y|\vec{x}'\rangle = y'|\vec{x}'\rangle$ ⋮

$z|\vec{x}'\rangle = z'|\vec{x}'\rangle$

$$1 = \int d^3x' \, |\vec{x}'\rangle\langle\vec{x}'| = \int d^3p' \, |\vec{p}'\rangle\langle\vec{p}'|$$

$$\langle\beta|\vec{p}|\alpha\rangle = \int d^3x' \, \psi_\beta^*(\vec{x}') \frac{\hbar}{i} \vec{\nabla} \psi_\alpha(\vec{x}')$$

$\rightarrow -i\hbar$

$$\langle\vec{x}'|\vec{p}'\rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}'\cdot\vec{x}'/\hbar}$$

$$\psi_\alpha(\vec{x}') = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3p' \, e^{i\vec{p}'\cdot\vec{x}'/\hbar} \phi_\alpha(\vec{p}')$$

$$\phi_\alpha(\vec{p}') = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3x' \, e^{-i\vec{p}'\cdot\vec{x}'/\hbar} \psi_\alpha(\vec{x}')$$