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Rotations in Quantum Mechanics

3D Rotation specified by 3x3 matrix R

R is orthogonal! $R^T R = \mathbb{1}$

$$\begin{bmatrix} v_x' \\ v_y' \\ v_z' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \quad \text{e.g. } \boxed{R_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Quantum Mechanics

$|\alpha\rangle_R = \mathcal{D}(R)\,|\alpha\rangle$ "rotated state"

- want $\mathcal{D}^\dagger(R)\,\mathcal{D}(R) = \mathbb{1}$
- Continuous operator: Rotate by angle ϕ about an axis $\hat{n}$

$$\boxed{\mathcal{D}(\hat{n}, d\phi) = 1 - i\,\frac{\vec{J}\cdot\hat{n}}{\hbar}\,d\phi}$$

$\vec{J} = J_x\,\hat{x} + J_y\,\hat{y} + J_z\,\hat{z}$ Three generators

Rotations do not commute!

$$R_z(\varepsilon) = \begin{bmatrix} 1-\frac{\varepsilon^2}{2} & -\varepsilon & 0 \\ \varepsilon & 1-\frac{\varepsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ε infinitesimal! taken to 2^{nd} order.

$$R_x(\varepsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\varepsilon^2}{2} & -\varepsilon \\ 0 & \varepsilon & 1-\frac{\varepsilon^2}{2} \end{bmatrix} \qquad R_y(\varepsilon) = \begin{bmatrix} 1-\frac{\varepsilon^2}{2} & 0 & \varepsilon \\ 0 & 1 & 0 \\ -\varepsilon & 0 & 1-\frac{\varepsilon^2}{2} \end{bmatrix}$$

$$R_x(\varepsilon) R_y(\varepsilon) = \begin{bmatrix} 1-\frac{\varepsilon^2}{2} & 0 & \varepsilon \\ \varepsilon^2 & 1-\frac{\varepsilon^2}{2} & -\varepsilon \\ -\varepsilon & \varepsilon & \boxed{\left(1-\frac{\varepsilon^2}{2}\right)^2} \end{bmatrix} \quad 1-\varepsilon^2$$

$$R_y(\varepsilon) R_x(\varepsilon) = \begin{bmatrix} 1-\frac{\varepsilon^2}{2} & \varepsilon^2 & \varepsilon \\ 0 & 1-\frac{\varepsilon^2}{2} & -\varepsilon \\ -\varepsilon & \varepsilon & 1-\varepsilon^2 \end{bmatrix}$$

NOT THE SAME!

$$R_x(\varepsilon)R_y(\varepsilon) - R_y(\varepsilon)R_x(\varepsilon) = \begin{bmatrix} 0 & -\varepsilon^2 & 0 \\ \varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= R_z(\varepsilon^2) - 1 = R_z(\varepsilon^2) - R_{any}(0)$$

$$\mathcal{D}(\hat{x},\varepsilon)\mathcal{D}(\hat{y},\varepsilon) - \mathcal{D}(\hat{y},\varepsilon)\mathcal{D}(\hat{x},\varepsilon)$$

$$= \mathcal{D}(\hat{z},\varepsilon^2) - \mathcal{D}(\hat{n},0)$$

$$\left[1 - \frac{i}{\hbar}J_x\varepsilon\right]\left[1 - \frac{i}{\hbar}J_y\varepsilon\right]$$

$$- \left[1 - \frac{i}{\hbar}J_y\varepsilon\right]\left[1 - \frac{i}{\hbar}J_x\varepsilon\right] = \left[1 - \frac{i}{\hbar}J_z\varepsilon^2\right] - 1$$

$$-\frac{i}{\hbar}J_y\varepsilon - \frac{i}{\hbar}J_x\varepsilon - \frac{1}{\hbar^2}J_xJ_y\varepsilon^2$$

$$- \left[-\frac{i}{\hbar}J_x\varepsilon - \frac{i}{\hbar}J_y\varepsilon - \frac{1}{\hbar^2}J_yJ_x\varepsilon^2\right] = -\frac{i}{\hbar}J_z\varepsilon^2$$

$$J_xJ_y - J_yJ_x = \hbar i J_z$$

$$\text{4D} \quad [J_x, J_y] = i\hbar J_z$$
$$\text{also} \quad [J_y, J_z] = i\hbar J_x$$
$$\text{\&} \quad [J_z, J_x] = i\hbar J_y$$

Angular Momentum commutation relations

$\vec{J} \equiv$ "Angular Momentum"

"Generator of rotations"

We write $\boxed{[J_i, J_j] = i\hbar\, \varepsilon_{ijk} J_k}$ Implied sum

$i, j, k = \{1, 2, 3\} = \{x, y, z\}$

$\varepsilon_{ijk} = 0$ if any two indices equal

$\varepsilon_{123} \equiv 1$ and flip any indices $\Rightarrow \times(-1)$

$= \varepsilon_{312} = \varepsilon_{231}$

$\varepsilon_{213} = -1 = \varepsilon_{321} = \varepsilon_{132}$

Finite Rotations

$$\mathcal{D}_z(\phi) = \lim_{N\to\infty}\left[1 - \frac{i}{\hbar} J_z\left(\frac{\phi}{N}\right)\right]^N$$

$$= \exp\left[-\frac{i}{\hbar} J_z \phi\right]$$

$$= 1 - \frac{i}{\hbar} J_z \phi + \frac{1}{2!}\left(-\frac{i}{\hbar}\right)^2 J_z^2 \phi^2 + \cdots$$

But $\mathcal{D}(\hat{n},\phi) \neq \exp\left[-\frac{i}{\hbar} \vec{J}\cdot\hat{n}\,\phi\right]$

Plausibility Check : Rotate "J_x" about z-axis

i.e. what is ${}_R\langle\alpha|J_x|\alpha\rangle_R$ $\quad |\alpha\rangle_R = \mathcal{D}_z(\phi)\,|\alpha\rangle$

$${}_R\langle\alpha|J_x|\alpha\rangle_R = \langle\alpha|\mathcal{D}_z^\dagger(\phi)\, J_x\, \mathcal{D}_z(\phi)|\alpha\rangle$$

$$= \langle\alpha|\, \underbrace{\exp\left[+\frac{i}{\hbar}J_z\phi\right] J_x \exp\left[-\frac{i}{\hbar}J_z\phi\right]}\, |\alpha\rangle$$

$$\left[1+\frac{i}{\hbar}J_z\phi+\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2 J_z^2\phi^2+\cdots\right]J_x$$

$$\left[1-\frac{i}{\hbar}J_z\phi+\frac{1}{2!}\left(-\frac{i}{\hbar}\right)^2 J_z^2\phi^2+\cdots\right]$$

$$=J_x+\frac{i}{\hbar}(J_zJ_x-J_xJ_z)\phi$$

$$+\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2(J_z^2J_x+J_xJ_z^2)\phi^2$$

$$+\left(\frac{i}{\hbar}\right)\left(\frac{-i}{\hbar}\right)\underbrace{J_zJ_xJ_z}_{\times 2}\phi^2+\cdots$$

$$J_zJ_x-J_xJ_z=[J_z,J_x]=i\hbar J_y$$

$$J_z^2J_x+J_xJ_z^2-2J_zJ_xJ_z$$

$$=J_z^2J_x-J_zJ_xJ_z+J_xJ_z^2-J_zJ_xJ_z$$

$$=J_z(J_zJ_x-J_xJ_z)+(J_xJ_z-J_zJ_x)J_z$$

$$=i\hbar J_zJ_y-i\hbar J_yJ_z=i\hbar[J_z,J_y]$$

$$=i\hbar(-i\hbar J_x)=\hbar^2J_x$$

$${}_R\langle\alpha|J_x|\alpha\rangle_R=\langle\alpha|\left[J_x+\frac{i}{\hbar}(i\hbar J_y)\phi\right.$$

$$\left.-\frac{1}{2!}\frac{1}{\hbar^2}\hbar^2J_x\phi^2+\cdots\right]|\alpha\rangle$$

$$= \langle\alpha|J_x|\alpha\rangle \left(1 - \frac{1}{2!}\phi^2 + \cdots\right)$$

$$- \langle\alpha|J_y|\alpha\rangle \left(\phi + \cdots\right)$$

$$= \langle\alpha|J_x|\alpha\rangle \cos\phi - \langle\alpha|J_y|\alpha\rangle \sin\phi$$

Right!

$$S_x = \frac{\hbar}{2}\left\{ |+\rangle\langle-| + |-\rangle\langle+| \right\}$$

$$S_y = i\frac{\hbar}{2}\left\{ -|+\rangle\langle-| + |-\rangle\langle+| \right\}$$

$$S_x S_y = i\frac{\hbar^2}{4}\left\{ 0 + |+\rangle\langle+| - |-\rangle\langle-| + 0 \right\}$$

$$S_y S_x = i\frac{\hbar^2}{4}\left\{ 0 - |+\rangle\langle+| + |-\rangle\langle-| + 0 \right\}$$

$$[S_x, S_y] = i\hbar \boxed{\frac{\hbar}{2}\left\{ |+\rangle\langle+| - |-\rangle\langle-| \right\}}$$

$$= i\hbar S_z$$

Angular momentum!