Phys 5701 13 Oct 2020

Today: "Choice of Gauge" in Quantum Mechanics

Final class on "Quantum Dynamics"

Thursday will start "Angular Momentum"

Reminder from Classical Physics

$$V(\vec{x}) \longrightarrow \tilde{V}(\vec{x}) = V(\vec{x}) + V_0$$

$$m\vec{a} = \vec{F} = -\vec{\nabla}V = -\vec{\nabla}\tilde{V}$$

No change in physics!

Big Effect in Quantum Mechanics!

$$|\alpha, t_0; t\rangle = e^{-iH(t-t_0)/\hbar}\,|\alpha, t_0\rangle$$

$$\longrightarrow \widetilde{|\alpha, t_0; t\rangle} = e^{-i\tilde{H}(t-t_0)/\hbar}\,|\alpha, t_0\rangle$$

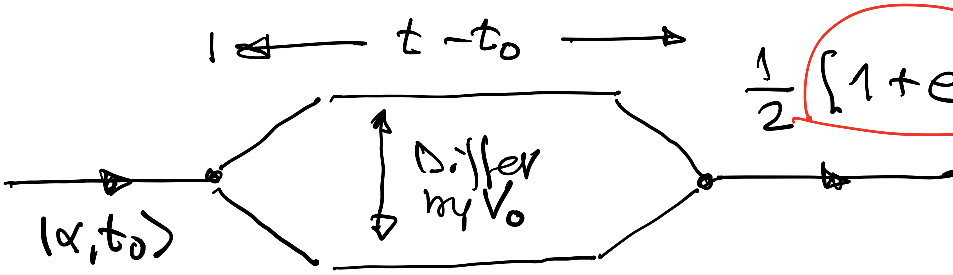
$$= \boxed{e^{-iV_0(t-t_0)/\hbar}}\,|\alpha, t_0; t\rangle$$

$$= e^{-i\phi(t)/\hbar}\,|\alpha, t_0; t\rangle$$

But $\langle\beta|\widetilde{\alpha, t_0; t}\rangle = e^{-i\phi/\hbar}\langle\beta|\alpha, t_0; t\rangle$

$$\Rightarrow \quad |\langle\beta|\widetilde{\alpha, t_0; t}\rangle|^2 = |\langle\beta|\alpha, t_0; t\rangle|^2$$

However consider...



$\frac{1}{2}[1+e^{i\phi}]\,|\alpha, t_0; t\rangle$
$\equiv |\alpha'\rangle$

$$\Rightarrow \quad |\langle\beta|\alpha'\rangle|^2 = \frac{1}{4}|1+e^{i\phi}|^2\,|\langle\beta|\alpha, t_0; t|^2$$

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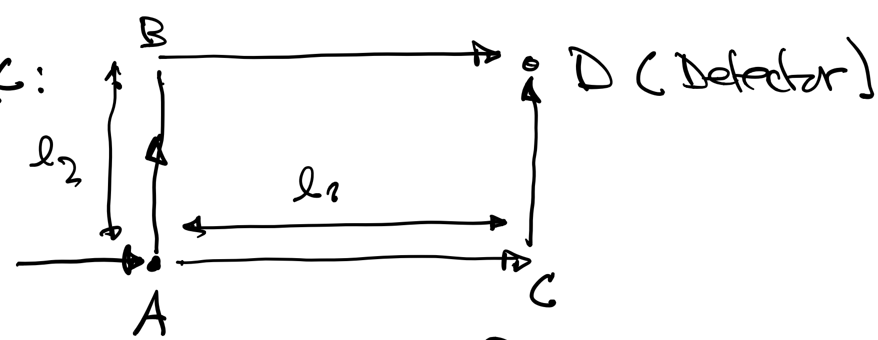
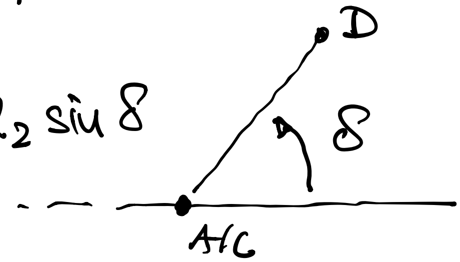
i.e. Probability oscillates as $\frac{1}{2}(1+\cos\phi(t))$
with $\phi(t) = V_0(t-t_0)/\hbar$

SEE PRL **34** (1975) 1472 POSTED!

SCHEMATIC:



Height $= mgl_2 \sin\delta$



Review of Electromagnetism   App. A, C

Maxwells Equations   in CGS UNITS!

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\,\partial\vec{B}/\partial t$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Lorentz Force Law

$$\vec{F} = q\vec{E} + \frac{1}{c}q\vec{v}\times\vec{B}$$

$$\vec{v} = \dot{\vec{x}}$$

Potentials (i.e. "electromagnetic potentials)

$$\vec{B}(\vec{x},\cancel{t}) = \vec{\nabla}\times\vec{A}(\vec{x},\cancel{t})$$

$$\vec{E}(\vec{x},\cancel{t}) = -\vec{\nabla}\phi(\vec{x},\cancel{t}) - \cancel{\frac{1}{c}\frac{\partial\vec{A}}{\partial t}}$$

i.e. Discuss $\vec{A}$ and ϕ not $\vec{E}$ and $\vec{B}$

STATIC.

Lagrangian

$$L(\vec{x},\dot{\vec{x}}) = \frac{1}{2}m\dot{\vec{x}}^2 - q\phi(\vec{x}) + \frac{q}{c}\dot{\vec{x}}\cdot\vec{A}(\vec{x})$$

Canonical Momentum

$$\vec{p} = \partial L/\partial\dot{\vec{x}} = m\dot{\vec{x}} + \frac{q}{c}\vec{A}(\vec{x})$$

Hamiltonian

$$\boxed{\begin{aligned} H(\vec{x},\vec{p}) &\equiv \dot{\vec{x}}\cdot\vec{p} - L(\vec{x},\dot{\vec{x}}(\vec{p})) \\ &= \frac{1}{2m}\left[\vec{p} - \frac{q}{c}\vec{A}(\vec{x})\right]^2 + q\phi(\vec{x}) \end{aligned}}$$

Electric Force: $q\phi(\vec{x}) = V(\vec{x})$

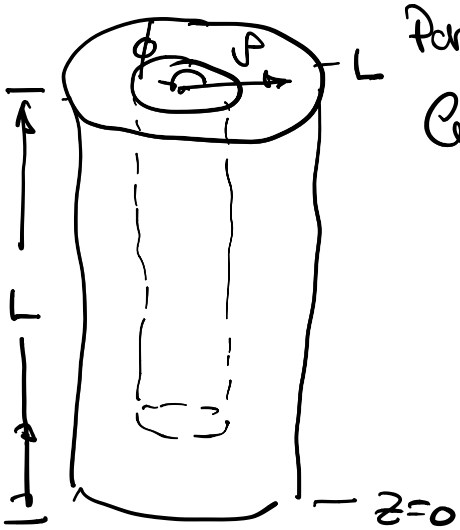
↳ Same as for "gravity"

Magnetic Field: More interesting!

$$\vec{A}(\vec{x}) \to \tilde{\vec{A}}(\vec{x}) = \vec{A}(\vec{x}) + \vec{\nabla}\Lambda(\vec{x})$$

↳ $\tilde{\vec{B}} = \vec{\nabla}\times\tilde{\vec{A}} = \underbrace{\vec{\nabla}\times\vec{A}}_{\vec{B}} + \underbrace{\vec{\nabla}\times\vec{\nabla}\Lambda}_{\equiv 0}$

Flux Quantization Prob. 2.37



Particle mass μ

Cylindrical coordinates (ρ, ϕ, z)

Particle confined to

$$\rho_a \le \rho \le \rho_b$$

"Free Particle"

$$\boxed{-\frac{\hbar^2}{2\mu}\vec{\nabla}^2\psi(\rho,\phi,z) = E\psi(\rho,\phi,z)}$$

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{z}\frac{\partial}{\partial z} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}$$

$$\vec{\nabla}^2 = \vec{\nabla}\cdot\vec{\nabla}$$

$$\text{So } \vec{\nabla}^2\psi = \vec{\nabla}\cdot\vec{\nabla}\psi$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial^2\psi}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}$$

- Separate variables: $\psi = R(\rho)\,\Phi(\phi)\,Z(z)$
- Boundary conditions

$$R(\rho_a) = R(\rho_b) = 0$$
$$Z(0) = 0 = Z(L)$$
$$\Phi(\phi + 2\pi) = \Phi(\phi)$$

- Write $E = \hbar^2 k_z^2 / 2\mu$

For homework, show that

$$\Phi(\phi) = e^{\pm i m \phi} \qquad m = 0, 1, 2, 3, \ldots$$

$$Z(z) = e^{\pm i \alpha_\ell z} \qquad \alpha_\ell \equiv \frac{\pi}{L}\ell \quad \ell = 1, 2, 3, \ldots$$

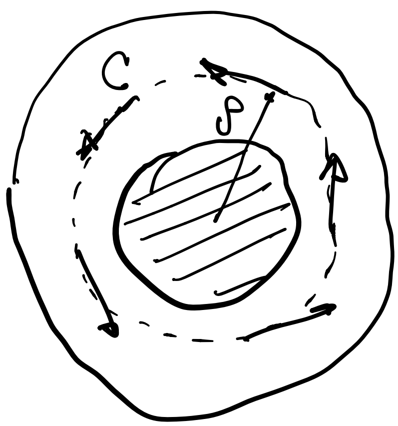
and $$\boxed{\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \left(\underbrace{k^2 - \alpha_\ell^2}_{} - \frac{m^2}{\rho^2}\right) R(\rho) = 0}$$

$\Rightarrow$ Bessel's Equation for $x = \kappa\rho$

$$\kappa^2 \equiv k^2 - \alpha_\ell^2$$

Then "Eigenvalues" for energy.

Now add $\vec{B}$ field: $\quad \rho \leq \rho_a$



$$\oint_C \vec{A}\cdot d\vec{\ell} = A\,(2\pi\rho)$$

$$= \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$$

$$= \int_S \vec{B}\cdot d\vec{S} = B\pi\rho_a^2$$

$$\Longrightarrow \vec{A} = \vec{A}(\rho) = \frac{B\rho_a^2}{2\rho}\,\hat{\underset{\sim}{\phi}}$$

"This effects things"

$$\underset{\sim}{\vec{p}} \to \vec{p} - \frac{e}{c}\underset{\approx}{\vec{A}} \qquad q = e < 0$$

$$\text{i.e. } \frac{\partial}{\partial\phi} \to \frac{\partial}{\partial\phi} - \underset{\sim\sim\sim\sim}{\frac{ie}{\hbar c}\frac{B\rho_a^2}{2}}$$

Using Gauge Transformations

i.e. $\vec{A} \longrightarrow \tilde{\vec{A}} = \underline{\underline{A}} + \vec{\nabla}\Lambda = 0$

where $\Lambda(\phi) = -\underbrace{B\rho_a^2}\phi/2$

Book: $\vec{A} \to \tilde{\vec{A}} \Rightarrow |\alpha\rangle \to |\tilde{\alpha}\rangle = G|\alpha\rangle$

$G^{+}G = 1$

want $\langle\alpha|\left[\vec{p} - \frac{e\vec{A}}{c}\right]|\alpha\rangle = \langle\tilde{\alpha}|\left[\vec{p} - \frac{e\tilde{\vec{A}}}{c}\right]|\tilde{\alpha}\rangle$

$\Leftrightarrow G = G(\vec{x}) = \exp\left[\frac{ie}{\hbar c}\Lambda(\vec{x})\right]$

For our problem

$G = G(\phi) = e^{-ig\phi}$

$g = eB\rho_a^2/2\hbar c$

$\Leftrightarrow \Phi(\phi) = e^{i\gamma\phi} \qquad \gamma = m \pm g$

$\Rightarrow g = \pm m \Rightarrow \boxed{B\pi\rho_a^2 = \pm\frac{hc}{e}m}$ 4×10^{-7} gauss$\cdot$cm^2