Phys 5701 12 Nov 2020

From Last Class: Central Potential Problems

Find eigenvalues and eigenfunctions for

$$\boxed{H = \frac{1}{2m}\vec{P}^2 + V(|\vec{x}|)} \qquad |\vec{x}| \equiv \sqrt{\vec{x}^2}$$

Saw that $\psi_{E\ell m}(\vec{x}) = R_{E\ell}(r)\, Y_\ell^m(\theta,\phi)$

Recall $\int \sin\theta\, d\theta\, Y_\ell^{m*} Y_\ell^m = 1$

$$\Rightarrow \int r^2 dr\, R_{E\ell}^* R_{E\ell} = 1$$

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left[r^2\frac{dR_{E\ell}}{dr}\right] + \frac{\ell(\ell+1)\hbar^2}{2mr^2}R_{E\ell}(r)$$

$$+V(r)\,R_{E\ell}(r) = E\,R_{E\ell}(r)$$

We studied $V(r)=0$

$$\Rightarrow R_{k\ell}(r) = N\, j_\ell(kr) \qquad E = \frac{\hbar k^2}{2m}$$

General Properties of $R_{E\ell}(r)$

Write $\boxed{R_{E\ell}(r) = u_{E\ell}(r) / r}$

NOTE: $\int r^2 dr\, R^2_{E\ell}(r) = 1 = \int dr\, u^2_{E\ell}(r)$

Now: $\frac{dR_{E\ell}}{dr} = \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2} u$

$$\frac{d}{dr}\left[r^2 \frac{dR_{E\ell}}{dr}\right] = \frac{du}{dr} + r\frac{d^2u}{dr^2} - \frac{du}{dr} = r\frac{d^2u}{dr^2}$$

$$-\frac{\hbar^2}{2m\cancel{r^2}}\,\cancel{r}\frac{d^2u}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\,\frac{u(r)}{\cancel{r}}$$

$$+ V(r)\frac{u(r)}{\cancel{r}} = E\,\frac{u(r)}{\cancel{r}}$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2u_{E\ell}}{dr^2} + \underbrace{\left[V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right]}_{V_{eff}(r)} u_{E\ell}(r) = E\,u_{E\ell}(r)}$$

"1D Schrödinger Eq. with $V \to V_{eff}$"

$\frac{\ell(\ell+1)\hbar^2}{2mr^2}$ = "centrifugal barrier"

Consider $r \to 0$: Centrifugal barrier dominates

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2mr^2}u = 0$$

$$\frac{d^2u}{dr^2} = \frac{\ell(\ell+1)}{r^2}u \qquad \text{Try } u(r) = r^s$$

$$s(s-1)\cancel{r^{s-2}} = \ell(\ell+1)\cancel{r^{s-2}}$$

Solve for s! $\quad s^2 - s - \ell(\ell+1) = 0$

$$s = \frac{1 \pm [1+4\ell^2+4\ell]^{1/2}}{2} = \frac{1 \pm [(1+2\ell)^2]^{1/2}}{2}$$

i.e. $s = 1+\ell$ or $s = -\ell$

For $r \to 0, \ldots$

BAD

$$u_{E\ell}(r) = Ar^{\ell+1} + \frac{B}{r^\ell}$$

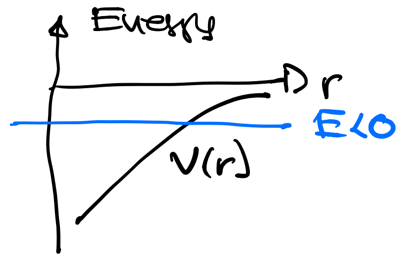
$\Rightarrow R_{E\ell}(r) = r^\ell$ for $r \to 0$

"$R_{E\ell}(r \to 0)$ appreciable only for s-states"

$\ell = 0, 1, 2, \ldots$

"s", "p", "d", ...

Consider $r \to \infty$: Depends on $V(r)$



$\boxed{\text{For } V(r) \to 0}$

and Bound states ($E<0$)

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = E\,u(r)$$

$$E \equiv -\frac{\hbar^2 k^2}{2m} \Rightarrow \frac{d^2u}{dr^2} = k^2 u(r)$$

$$\Rightarrow u(r) = e^{-kr} \text{ for } r \to \infty \qquad k \equiv (-2mE/\hbar^2)^{1/2}$$

$$\text{so } u(r) = (kr)^{\ell+1}\,\boxed{e^{-kr}}\,w(kr)$$

Put $\rho \equiv kr$ then ...

$$\boxed{\frac{d^2w}{d\rho^2} + 2\left[\frac{\ell+1}{\rho} - 1\right]\frac{dw}{d\rho} + \left[\frac{V}{E} - \frac{2(\ell+1)}{\rho}\right]w(\rho) = 0}$$

with $V = V(r = \rho/k)$

Application: Coulomb Potential

$$V(r) = -\frac{Ze^2}{r} \qquad \text{[See Appendix A]}$$

$$\frac{V}{E} = \left(\frac{1}{-E}\right)\frac{Ze^2}{\rho/\kappa} = \left(-\frac{1}{E}\right)\left[-\frac{2mE}{\hbar^2}\right]^{1/2}\frac{Ze^2}{\rho}$$

$$= \left[\frac{2m}{-E}\right]^{1/2}\frac{Ze^2}{\hbar\rho} = \underline{\underline{\left[\frac{2mc^2}{-E}\right]}}^{1/2}\frac{e^2}{\hbar c}\frac{Z}{\rho}$$

Dimensionless

$$\frac{e^2}{\hbar c} \equiv \alpha \quad \text{"Fine structure constant"}$$

$$\cong 1/137$$

$$\boxed{\rho_0 \equiv (2mc^2/-E)^{1/2} Z\alpha}$$

i.e. $\frac{V}{E} = \frac{\rho_0}{\rho}$

$$\boxed{\rho\frac{d^2w}{d\rho^2} + 2(\underline{\underline{\ell+1}}-\rho)\frac{dw}{d\rho} + [\underline{\underline{\rho_0 - 2(\ell+1)}}]\, w(\rho) = 0}$$

$$w(\rho) = a_0 + a_1\rho + a_2\rho^2 + \cdots = \sum_i a_i \rho^i$$

Kummer's Equation

$$x \frac{d^2F}{dx^2} + (c-x)\frac{dF}{dx} - a\,F(x) = 0$$

$$F(a;c;x) = 1 + \frac{a}{c}x + \frac{\boxed{a(a+1)}}{c(c+1)}\frac{x^2}{2!} + \cdots \quad *$$

Put $x = 2\rho$ and multiply by 2

$$\rho \frac{d^2F}{d\rho^2} + 2\left(\frac{c}{2} - \rho\right)\frac{dF}{d\rho} - 2a\,F(2\rho) = 0$$

$\Leftarrow$ $w(\rho) = F(a;c;2\rho)$

$$\frac{c}{2} = \ell+1 \qquad -2a = \rho_0 - 2(\ell+1) \leftarrow$$

F = "Confluent Hypergeometric Function"

For large ρ only highest powers contribute.

$$\frac{a(a+1)(a+2)\cdots(a+N-1)}{c(c+1)(c+2)\cdots(c+N-1)}\frac{(2\rho)^N}{N!} > \underbrace{\frac{a^N}{c^N}}_{\rightarrow \frac{1}{2^N}}\frac{(\cancel{2}\rho)^N}{N!}$$

$$\approx \frac{\rho^N}{N!} = e^{\rho} \;!!$$

$\Rightarrow$ u Grows too fast!!

$\hookrightarrow$ Series must terminate!

i.e. For some N $\quad a+N=0 \quad N=0,1,2,3,\dots$

$$a = \ell+1-\rho_0/2 = -N$$

$$\rho_0 = 2(N+\ell+1) \qquad N=0,1,2,\dots \quad \ell=0,1,2,\dots$$

$$= 2n \quad n \equiv N+\ell+1 = 1,2,3,\dots$$

$$\ell = 0,1,\dots,n-1$$

$$\frac{2mc^2}{-E} Z^2\alpha^2 = 4n^2 \Rightarrow \boxed{E = -\frac{1}{2}mc^2 \frac{Z^2\alpha^2}{n^2}}$$

$n=1$	$\ell=0$	1 state
$n=2$	$\ell=0\ (1),\ 1\ (3)$	4
$n=3$	$\ell=0\ (1),\ 1\ (3),\ 2\ (5)$	9

$\hookrightarrow$ State $|n\rangle$ has n^2 degeneracy!

Normalization, etc.., see Book.

Isotropic Harmonic Oscillator

$$V(r) = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2(x^2+y^2+z^2)$$