Phys 5701   10 Sep 2020

Today: Position and Momentum

Continuous Eigenvalues

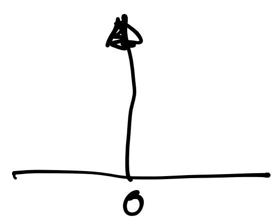
$$\xi|\xi'\rangle = \xi'|\xi'\rangle$$

Recall: $\langle a'|a''\rangle = \delta_{a''a'}$

Now: $\langle \xi'|\xi''\rangle = \delta(\xi'-\xi'')$

Dirac Delta Function

$\delta(x)=0$ for $x\neq 0$ but



$$\int_{-\infty}^{\infty}\delta(x)\,dx = 1 = \int_{-\varepsilon}^{\varepsilon}\delta(x)\,dx$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

$y \equiv k\varepsilon$

$$\int_{-\varepsilon}^{\varepsilon}\delta(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik\varepsilon}-e^{-ik\varepsilon}}{i(k\varepsilon)}\,d(k\varepsilon) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(y)}{y}\,dy$$

$$= 1$$

$\Rightarrow \quad \mathbb{1} = \int d\xi' \, |\xi'\rangle\langle\xi'|$

$$|\alpha\rangle = \int d\xi' \, |\xi'\rangle\langle\xi'|\alpha\rangle$$

$$\langle\alpha|\alpha\rangle = 1 = \int d\xi' \, |\langle\xi'|\alpha\rangle|^2$$

$$\langle\beta|\alpha\rangle = \int d\xi' \, \langle\beta|\xi'\rangle\langle\xi'|\alpha\rangle$$

The Position Operator (1D first)

$$x\,|x'\rangle = x'\,|x'\rangle$$

$$\langle x'|\alpha\rangle = \psi_\alpha(x')$$

$$\Rightarrow 1 = \int dx' \, |\langle x'|\alpha\rangle|^2$$

$$= \int dx' \, \underline{\underline{|\psi_\alpha(x')|^2}} = 1$$

"Normalized Wave Function"

Probability of finding "particle" "near" position x'

$$= \underbrace{|\langle x'|\alpha\rangle|^2}\Delta x'$$

$$|\psi_\alpha(x')|^2$$ "Probability Density"

<u>Generalize to 3D</u>

$\vec{x}$ = 3D position vector (NOT $\vec{r}$)

Eigenstate $|\vec{x}'\rangle = |x, y, z\rangle$

Assumption: $[x, y] = 0$ etc...

i.e. $[x_i, x_j] = 0 \quad i, j = 1, 2, 3$
(x, y, z)

$$\langle \vec{x}' | \vec{x}'' \rangle = \delta^{(3)}(\vec{x}' - \vec{x}'')$$
$$= \delta(x'-x'')\delta(y'-y'')\delta(z'-z'')$$

SLOPPY $\delta(\vec{x}' - \vec{x}'')$

Translation "change of basis"

i.e. UNITARY TRANSFORMATION

i.e. $U^{\dagger}U = \mathbb{1}$

BTW $\langle\alpha|\alpha\rangle = \langle\alpha|U^{\dagger}U|\alpha\rangle$

$= (\langle\alpha|U^{\dagger})(U|\alpha\rangle)$

"PRESERVES NORMALIZATION"

Translation Operator $\mathcal{J}(d\vec{x}')$

$$\mathcal{J}(d\vec{x}')|\vec{x}'\rangle = |\vec{x}'+d\vec{x}'\rangle$$

want $\mathcal{J}^{\dagger}(d\vec{x}')\mathcal{J}(d\vec{x}') = 1$

$$\mathcal{J}(-d\vec{x}')\mathcal{J}(d\vec{x}') = 1$$

$\Rightarrow \mathcal{J}(-d\vec{x}') = \mathcal{J}^{-1}(d\vec{x}') = \mathcal{J}^{\dagger}(d\vec{x}')$

also $\lim_{d\vec{x}'\to 0} \mathcal{J}(d\vec{x}') = 1$

and $\mathcal{J}(d\vec{x}'')\mathcal{J}(d\vec{x}''') = \mathcal{J}(d\vec{x}'+d\vec{x}'')$

$$\Rightarrow \boxed{\mathcal{J}(d\vec{x}') = 1 - i\vec{K}\cdot d\vec{x}'}$$

$\vec{K}$ = Hermitian Operator $= \vec{K}^\dagger$

e.g. $\mathcal{J}^\dagger(d\vec{x}') = 1 + i\vec{K}\cdot d\vec{x}' = \mathcal{J}(-d\vec{x}')$

$$\mathcal{J}^\dagger(d\vec{x}')\,\mathcal{J}(d\vec{x}') = 1 + \underline{\underline{\mathcal{O}\left((d\vec{x}')^2\right)}}$$

<u>NOMENCLATURE</u>: $(\hbar)\vec{K}$ is "generator" of translation

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Let's calculate $[\vec{x}, \mathcal{J}(d\vec{x}')]$

$$\vec{x}\,\mathcal{J}(d\vec{x}')|\vec{x}'\rangle = \vec{x}\,|\vec{x}'+d\vec{x}'\rangle$$
$$= (\vec{x}'+d\vec{x}')\,|\vec{x}'+d\vec{x}'\rangle$$

$$\mathcal{J}(d\vec{x}')\,\vec{x}\,|\vec{x}'\rangle = \vec{x}'\,\mathcal{J}(d\vec{x}')\,|\vec{x}'\rangle$$
$$= \vec{x}'\,|\vec{x}'+d\vec{x}'\rangle$$

$$\Rightarrow [\vec{x}, \mathcal{J}(d\vec{x}')]\,|\vec{x}'\rangle = d\vec{x}'\,|\vec{x}'+d\vec{x}'\rangle$$

FOR ARBITRARY $\langle\alpha|$ ...

$$\langle\alpha|[\vec{x}, \mathcal{J}(d\vec{x}')]|\vec{x}'\rangle = d\vec{x}'\,\langle\alpha|\vec{x}'+d\vec{x}'\rangle$$

$$= d\vec{x}'\left[\langle\alpha|\vec{x}'\rangle + \vec{\nabla}\langle\alpha|\vec{x}'\rangle\cdot d\vec{x}'\right]$$

SINCE $\langle \alpha |$ IS ARBITRARY,

$$\Leftrightarrow \quad [\vec{x}, \mathcal{J}(d\vec{x}')] = d\vec{x}' \, (\underline{1})$$

$$\mathcal{J}(d\vec{x}') = \underline{1} - i\vec{K} \cdot d\vec{x}'$$

$$[\vec{x}, \mathcal{J}(d\vec{x}')] = \vec{x} - i\vec{x}\,\vec{K} \cdot d\vec{x}' - \vec{x} + i\vec{K} \cdot d\vec{x}'\,\vec{x} = d\vec{x}'$$

$$\vec{x}\,\vec{K} \cdot d\vec{x}' - \vec{K} \cdot d\vec{x}'\,\vec{x} = i\,d\vec{x}'$$

VECTOR EQUATION

$$x_i K_j dx'_j - K_j dx'_j x_i = i\,dx'_i$$

$$(x_i K_j - K_j x_i)\,dx'_j = i\,dx'_j\,\delta_{ij}$$

$$\boxed{[x_i, K_j] = i\,\delta_{ij}}$$

"Canonical Commutation Relation"

i.e. $[x, K_x] = i$, $[y, K_z] = 0$ etc...

$\hbar \vec{K}$ = Generator of translations

$= \vec{p}$ momentum.

$\Leftrightarrow \quad [x_i, p_j] = i\hbar\,\delta_{ij}$

Recall "Uncertainty Relation"

$$\underbrace{\langle (\Delta A)^2 \rangle}\ \langle (\Delta B)^2 \rangle \geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2$$

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_x)^2 \rangle \geq \frac{1}{4} \hbar^2$$

$$\langle (\Delta x)^2 \rangle^{1/2} \langle (\Delta p_x)^2 \rangle^{1/2} \geq \frac{\hbar}{2}$$

PRECISELY DEFINED!

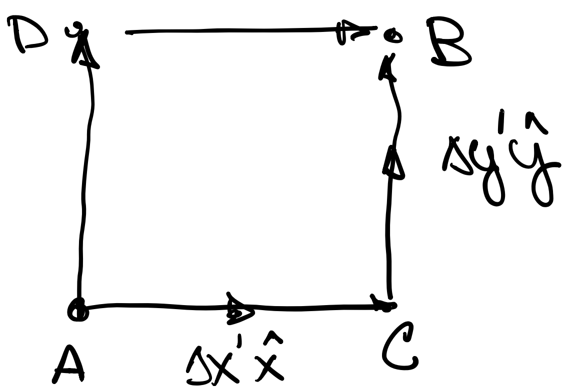
FINITE TRANSLATIONS

$$\mathcal{J}(\Delta x \hat{x}) = \lim_{N\to\infty}\left[1 - i\frac{p_x}{\hbar}\frac{\Delta x'}{N}\right]^N$$

$$= \exp\left[-i p_x \Delta x'/\hbar\right]$$

$$(1+\text{tiny})^{1/\text{tiny}} = 2.71828\ldots$$

YOU KNOW: FINITE TRANSLATIONS COMMUTE.



$$\mathcal{J}(\Delta y'\hat{y})\,\mathcal{J}(\Delta x'\hat{x}) = \mathcal{J}(\Delta x'\hat{x})\,\mathcal{J}(\Delta y'\hat{y})$$

$$\exp\left[-i\frac{P_y\Delta y'}{\hbar}\right]\exp\left[-i\frac{P_x\Delta x'}{\hbar}\right]$$

$$= \exp\left[-i\frac{P_x\Delta x'}{\hbar}\right]\exp\left[-i\frac{P_y\Delta y'}{\hbar}\right]$$

$$\left[1 - i\frac{P_y\Delta y'}{\hbar} - \frac{1}{2}\frac{P_y^2(\Delta y')^2}{\hbar^2} + \cdots\right]\left[\quad\right]$$

$$= \left[\quad\right]\left[\quad\right]$$

Keep all to second order

$$\left(-i\frac{P_y\underline{\underline{\Delta y'}}}{\hbar}\right)\left(-i\frac{P_x\underline{\underline{\Delta x'}}}{\hbar}\right)$$

$$= \left(-i\frac{P_x\underline{\underline{\Delta x'}}}{\hbar}\right)\left(-i\frac{P_y\underline{\underline{\Delta y'}}}{\hbar}\right)$$

$\Leftrightarrow P_yP_x = P_xP_y$ COMMUTE!

$[X_i, X_j] = 0 \quad \boxed{[X_i, P_j] = i\hbar\delta_{ij}}$

"Translations Commute" $\Rightarrow [P_i, P_j] = 0$

i.e. $|P_x, P_y, P_z\rangle$