Phys 5701 10 Nov 2020

Today: Central Potential Problems

Our Goal: $H|E\rangle = E|E\rangle$ via

$$\langle \vec{x}'|H|E\rangle = E\langle \vec{x}'|E\rangle$$

where we write $\psi_E(\vec{x}') = \langle \vec{x}'|E\rangle$

with $H = \frac{1}{2m}\vec{p}^{\,2} + V(|\vec{x}|)$ $\quad |\vec{x}| \equiv \sqrt{\vec{x}^{\,2}}$

"Spherically Symmetric Potentials"

Note: $[\vec{x}^{\,2}, \vec{L}] = 0 = [\vec{p}^{\,2}, \vec{L}]$

Easy to show this explicitly!

$x_n p_k - p_k x_n = i\hbar\,\delta_{kn}$

$$[x_n x_n, L_i] = x_n x_n \varepsilon_{ijk} x_j p_k - \varepsilon_{ijk} x_j p_k x_n x_n$$

$$= \varepsilon_{ijk} x_n x_n x_j p_k - \varepsilon_{ijk} x_j (x_n p_k - i\hbar\,\delta_{kn}) x_n$$

$$= \varepsilon_{ijk} x_n x_n x_j p_k - \varepsilon_{ijk} x_j x_n (x_n p_k - i\hbar\,\delta_{kn}) + i\hbar\,\varepsilon_{ijk} x_j x_k$$

$$= \varepsilon_{ijk} x_n x_n x_j p_k - \varepsilon_{ijk} x_j x_n x_n p_k + 2i\hbar\,\varepsilon_{ijk} x_j x_k$$

But (1) $x_n x_n x_j = x_j x_n x_n$

(2) $\varepsilon_{ijk} x_j x_k = (\vec{x} \times \vec{x})_i = 0$

$\Rightarrow [x_n x_n, L_i] = 0$

i.e. $[\vec{x}^2, \vec{L}] = 0$

Same for $\vec{p}^2$

$$\boxed{H = \frac{1}{2m}\vec{p}^2 + V(|\vec{x}|)} \qquad |\vec{x}| \equiv \sqrt{\vec{x}^2}$$

$\Rightarrow [\vec{L}, H] = 0$

"Angular Momentum is Conserved"

i.e. $\frac{d\vec{L}}{dt} = \frac{1}{i\hbar}[\vec{L}, H] = 0$

also $[H, \vec{L}^2] = 0 = [H, L_z]$

$|E\rangle \longrightarrow |E \ell m\rangle$

"Orbital Angular Momentum is a Good Quantum Number"

Recall that

$$\vec{L}^2 = \vec{x}^2\vec{p}^2 - (\vec{x}\cdot\vec{p})^2 + i\hbar\,\vec{x}\cdot\vec{p} \qquad (3.226)$$

want to write $\underbrace{\langle\vec{x}'|H|Elm\rangle}$ $= E\,\psi_{Elm}(\vec{x}')$

i.e. $\langle\vec{x}'|\vec{L}^2|Elm\rangle$ (1)

$= \langle\vec{x}'|\vec{x}^2\vec{p}^2|Elm\rangle$ (2)

$- \langle\vec{x}'|(\vec{x}\cdot\vec{p})^2|Elm\rangle$ (4)

$+ i\hbar\,\langle\vec{x}'|\vec{x}\cdot\vec{p}|Elm\rangle$ (3)

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(1) $\langle\vec{x}'|\vec{L}^2|Elm\rangle = l(l+1)\hbar^2\underbrace{\langle\vec{x}'|Elm\rangle}$

(2) $\langle\vec{x}'|\vec{x}^2\vec{p}^2|Elm\rangle = \vec{x}'^2\langle\vec{x}'|\vec{p}^2|Elm\rangle$

$= r^2\langle\vec{x}'|\vec{p}^2|Elm\rangle$

(3) $\langle\vec{x}'|\vec{x}\cdot\vec{p}|Elm\rangle = \vec{x}'\cdot\underbrace{\langle\vec{x}'|\vec{p}|Elm\rangle}$

$= \vec{x}'\cdot(-i\hbar\vec{\nabla}')\langle\vec{x}'|Elm\rangle$

But $\vec{x}' = r\hat{r}$

$\Rightarrow \langle \vec{x}' | \vec{x}\cdot\vec{p} | E\ell m\rangle$

$$= -i\hbar\, r \frac{\partial}{\partial r} \langle \vec{x}' | E\ell m\rangle$$

(4) $\langle \vec{x}' | (\vec{x}\cdot\vec{p})^2 | E\ell m\rangle$

$$= -i\hbar\, r \frac{\partial}{\partial r} \langle \vec{x}' | \vec{x}\cdot\vec{p} | E\ell m\rangle$$

$$= -\hbar^2\, r \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} \langle \vec{x}' | E\ell m\rangle \right]$$

$\langle \vec{x}' | \vec{L}^2 | E\ell m\rangle$ (1)

$= \langle \vec{x}' | \vec{x}^2 \vec{p}^2 | E\ell m\rangle$ (2)

$- \langle \vec{x}' | (\vec{x}\cdot\vec{p})^2 | E\ell m\rangle$ (4)

$+ \underline{i\hbar} \langle \vec{x}' | \vec{x}\cdot\vec{p} | E\ell m\rangle$ (3) Becomes...

$$\boxed{\begin{aligned} &\ell(\ell+1)\hbar^2\, \psi_{E\ell m}(r,\theta,\phi) \\ &= \underline{r^2} \langle \vec{x}' | \vec{p}^2 | E\ell m\rangle \\ &\quad + \hbar^2\, r \frac{\partial}{\partial r}\left[r \frac{\partial}{\partial r} \psi_{E\ell m}(r,\theta,\phi)\right] \\ &\quad + \hbar^2\, r \frac{\partial}{\partial r} \psi_{E\ell m}(r,\theta,\phi) \end{aligned}}$$

$$\underbrace{\langle \vec{x}' | H | E \rangle} = \frac{1}{2m} \langle \vec{x}' | \vec{p}^2 | E l m \rangle + \langle \vec{x}' | \underbrace{V(|\vec{x}|)} | E l m \rangle$$

$$= \boxed{\begin{aligned} &-\frac{\hbar^2}{2m}\frac{1}{r^2}\left[r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r}\right]\psi_{Elm}(r,\theta,\phi) \\ &+ \frac{l(l+1)\hbar^2}{2mr^2}\psi_{Elm}(r,\theta,\phi) \\ &+ V(r)\psi_{Elm}(r,\theta,\phi) \\ &= E\psi_{Elm}(r,\theta,\phi) \end{aligned}}$$

Only radial derivatives!

i.e. $\langle \vec{x}' | \vec{L}^2 | l m \rangle = l(l+1)\hbar^2 Y_l^m(\theta,\phi)$

$\Leftarrow\!\triangleright$ $\psi_{Elm}(r,\theta,\phi) = R_{El}(r)\, Y_l^m(\theta,\phi)$

$$\boxed{\begin{aligned} &-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left[r^2\frac{dR_{El}}{dr}\right] + \underline{\frac{l(l+1)\hbar^2}{2mr^2}}R_{El}(r) \\ &+ V(r)R_{El}(r) = E R_{El}(r) \end{aligned}}$$

Example: $V(r) = 0$ "Free particle"

$$E = \frac{\hbar k^2}{2m} \qquad \rho \equiv kr \Rightarrow \frac{1}{dr} = \frac{k}{d\rho}$$

$$\Rightarrow -\frac{\cancel{\hbar^2}}{\cancel{2m}} \frac{\cancel{k^2}}{\rho^2} \cancel{k^2} \frac{d}{d\rho}\left[\frac{\rho^2}{\cancel{k^2}} \frac{d}{d\rho} R(\rho)\right]$$

$$+ \frac{\ell(\ell+1)}{\cancel{2m}} \frac{\cancel{\hbar^2}}{\rho^2} \cancel{k^2} R(\rho) = \frac{\hbar^2 \cancel{k^2}}{2m} R(\rho)$$

$$\frac{1}{\rho^2} \frac{d}{d\rho}\left[\rho^2 \frac{dR}{d\rho}\right] - \frac{\ell(\ell+1)}{\rho^2} R(\rho) + R(\rho) = 0$$

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho} \frac{dR}{d\rho} + \left[1 - \frac{\ell(\ell+1)}{\rho^2}\right] R(\rho) = 0$$

$$\ell = 0, 1, 2, 3, \ldots$$

Solutions "Spherical Bessel Functions"

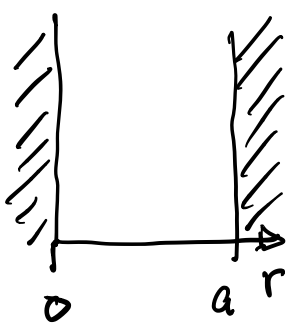
$$j_\ell(\rho) = (-\rho)^\ell \left[\frac{1}{\rho} \frac{d}{d\rho}\right]^\ell \left[\frac{\sin \rho}{\rho}\right]$$

$$n_\ell(\rho) = -(-\rho)^\ell \left[\frac{1}{\rho} \frac{d}{d\rho}\right]^\ell \left[\frac{\cos \rho}{\rho}\right]$$

Application: Spherical Infinite Well

i.e. $V(r) = 0$ for $r \leq a$

$V(r) \to \infty$ for $r > a$



$$R_{El}(r) = N j_l(kr)$$

with $j_l(ka) = 0$

For $l=0$ $\quad R_{El=0}(r) = N j_0(kr)$

$$j_0(kr) = \frac{\sin(kr)}{kr}$$

i.e. $\sin(ka) = 0$

$$\Rightarrow ka = \pi, 2\pi, 3\pi, \ldots$$

$$E_{l=0} = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2ma^2}\left[\pi^2, (2\pi)^2, (3\pi)^2, \ldots\right]$$

[Fig. 3.6]

For $l=1$, $j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}$ (B.41)