Phys 5701 8 Sep 2020

Today: "Change of Basis"

Unitary Operators & Diagonalization

Two Observables A & B w/ $[A, B] \neq 0$

↳ Can consider basis states

$\{|a'\rangle\}$ OR $\{|b'\rangle\}$

Example: $A = S_z$ $\{|a'\rangle\} = |+\rangle, |-\rangle$

$B = S_x$ $\{|b'\rangle\} = |S_x;+\rangle, |S_x;-\rangle$

Theorem: Operator U with

$$U|a^{(1)}\rangle = |b^{(1)}\rangle \cdots U|a^{(N)}\rangle = |b^{(N)}\rangle$$

where U is "Unitary" Operator

$$U^{\dagger}U = 1 = UU^{\dagger} \quad \text{i.e. } U^{\dagger} = U^{-1}$$

<u>Proof</u>: $U = \sum_k |b^{(k)}\rangle\langle a^{(k)}|$ <u>works!</u>

$$U|a^{(l)}\rangle = \sum_k |b^{(k)}\rangle \underbrace{\langle a^{(k)}|a^{(l)}\rangle}$$

$$= |b^{(l)}\rangle \quad \text{RIGHT!}$$

$$U^+ = \sum_k |a^{(k)}\rangle\langle b^{(k)}|$$

$$U^+U = \sum_k \sum_l |a^{(k)}\rangle \underbrace{\langle b^{(k)}|b^{(l)}\rangle}_{\delta_{kl}} \langle a^{(l)}|$$

$$= \sum_k |a^{(k)}\rangle\langle a^{(k)}| = 1 \quad \text{RIGHT.}$$

Consider U represented in $\{|a'\rangle\}$ basis

$$\langle a^{(k)}|U|a^{(l)}\rangle = \langle a^{(k)}|b^{(l)}\rangle$$

"Transformation matrix"

SEE (1.149)

Suppose know expansion of $|\alpha\rangle$ in $\{|a'\rangle\}$.

i.e. $|\alpha\rangle = \sum_{a'} |a'\rangle \langle a'|\alpha\rangle$

$$= \sum_{l} |a^{(l)}\rangle \langle a^{(l)}|\alpha\rangle$$

But want $|\alpha\rangle = \sum_{k} |b^{(k)}\rangle \langle b^{(k)}|\alpha\rangle$

Easy! Insert complete set of states!

$$\langle b^{(k)}|\alpha\rangle = \sum_{l} \langle b^{(k)}|a^{(l)}\rangle \langle a^{(l)}|\alpha\rangle$$

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But $\langle b^{(k)}|a^{(l)}\rangle = \langle a^{(k)}|U^{\dagger}|a^{(l)}\rangle$

$$\Rightarrow \underbrace{\langle b^{(k)}|\alpha\rangle}_{\text{column vector}} = \sum_{l} \underbrace{\langle a^{(k)}|U^{\dagger}|a^{(l)}\rangle}_{\text{matrix}} \underbrace{\langle a^{(l)}|\alpha\rangle}_{\text{column vector}}$$

<u>Example</u>: Have $|S_x;+\rangle$ $(=|\alpha\rangle)$

$$\boxed{= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle}$$

i.e. $\{|a'\rangle\} = |+\rangle, |-\rangle$ "old basis"

"New Basis": $\{|b'\rangle\} = |S_x;+\rangle, |S_x;-\rangle$

Construct $U = |S_x;+\rangle\langle+| + |S_x;-\rangle\langle-|$

$$= \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right]\langle+| + \frac{1}{\sqrt{2}}\left[|+\rangle - |-\rangle\right]\langle-|$$

$$\boxed{\begin{aligned} &= \frac{1}{\sqrt{2}}|+\rangle\langle+| + \frac{1}{\sqrt{2}}|-\rangle\langle+| \\ &\quad + \frac{1}{\sqrt{2}}|+\rangle\langle-| - \frac{1}{\sqrt{2}}|-\rangle\langle-| \end{aligned}}$$

Construct "$\langle a^{(k)} | U^{+} | a^{(l)} \rangle$"

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \equiv U^{+}_{kl}$$

So what are the coefficients of $|\alpha\rangle \equiv |S_x;+\rangle$ in $\{|b'\rangle\} = \{|S_x;\pm\rangle\}$ basis?

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ RIGHT!

TRY THIS: Express $|\alpha\rangle = |S_y;+\rangle$ in the $|S_x;\pm\rangle$ basis.

What about matrix transformations?

$$\langle b^{(k)} | X | b^{(l)} \rangle$$

$$= \sum_m \sum_n \underbrace{\langle b^{(k)} | a^{(m)} \rangle}\langle a^{(m)} | X | a^{(n)} \rangle \underbrace{\langle a^{(n)} | b^{(l)} \rangle}$$

$$= \sum_m \sum_n \langle a^{(k)} | U^{\dagger} | a^{(m)} \rangle \langle a^{(m)} | X | a^{(n)} \rangle \langle a^{(n)} | U | a^{(l)} \rangle$$

i.e. $X' = U^{\dagger} X U$ similarity transformation.

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the trace of an operator

$$\mathrm{Tr}(X) \equiv \sum_{a'} \langle a' | X | a' \rangle$$

$$\sum_{a'} \langle a' | X | a' \rangle = \sum_{a'} \sum_{b'} \sum_{b''} \langle a' | b' \rangle \langle b' | X | b'' \rangle \langle b'' | a' \rangle$$

$$= \sum_{b'} \sum_{b''} \sum_{a'} \underbrace{\langle b'' | a' \rangle \langle a' | b' \rangle} \langle b' | X | b'' \rangle$$

$$= \sum_{b'} \sum_{b''} \underbrace{\langle b'' | b' \rangle} \langle b' | X | b'' \rangle = \sum_{b'} \langle b' | X | b' \rangle$$

TRACE IS REPRESENTATION INDEPENDENT

Diagonalization

want to solve $B|b'\rangle = b'|b'\rangle$

when we have $\langle a''|B|a'\rangle$ in $\{|a'\rangle\}$

$$\sum_{a'} B|a'\rangle\langle a'|b'\rangle = b'|b'\rangle$$

$$\Leftrightarrow \sum_{a'} \langle a''|B|a'\rangle\langle a'|b'\rangle = b'\langle a''|b'\rangle$$

$$\begin{bmatrix} B_{11} & B_{12} & \cdots \\ B_{21} & B_{22} & \cdots \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} c_1^{(l)} \\ c_2^{(l)} \\ \vdots \end{bmatrix} = \underset{(\text{aka } \lambda)}{b^{(l)}} \begin{bmatrix} c_1^{(l)} \\ c_2^{(l)} \\ \vdots \end{bmatrix}$$

$$c_k^{(l)} \equiv \langle a^{(k)}|b^{(l)}\rangle$$

"Eigenvalue problem"

$$\underline{\underline{B}}\,\underline{b} = \lambda\,\underline{b} \Rightarrow \text{Find } b\text{'s and } \lambda\text{'s}$$

$$= \lambda\,\underline{\underline{1}}\,\underline{b}$$

i.e. $(\underline{\underline{B}} - \lambda\,\underline{\underline{1}})\,\underline{b} = 0 \;{*} \rightarrow \underline{b}\text{'s}$

$\Leftrightarrow \det(\underline{\underline{B}} - \lambda\,\underline{\underline{1}}) = 0 \Rightarrow \lambda\text{'s}$

<u>Example</u>: Find eiv's of S_y in $|\pm\rangle$ basis

$$S_y = \frac{\hbar}{2}\left[-i\,\underbrace{|+\rangle\langle-| + i\,|-\rangle\langle+|}\right] \qquad (1.96b)$$

$$S_y \doteq \text{"}B\text{"} = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\boxed{\frac{\hbar}{2}\underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} a \\ b \end{bmatrix}}$$

$$\text{Det}\begin{bmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 - \left(-i\frac{\hbar}{2}\right)\left(i\frac{\hbar}{2}\right)$$

$$= \lambda^2 - (\hbar/2)^2 = 0$$

$\Rightarrow$ Eigenvalues are $+\frac{\hbar}{2}$ $-\frac{\hbar}{2}$ RIGHT.

Eigenvector for $+\hbar/2$: $-\hbar/2$

$$\begin{bmatrix} +\hbar/2 & -i\hbar/2 \\ i\hbar/2 & +\hbar/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} +a - ib &= 0 \text{ (top)} \\ ia + b &= 0 \text{ (bottom)} \end{aligned}\right\} \text{ same equation!!}$$

$\Rightarrow$ $b = ia$ and then normalize!

So $|S_y; +\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle$ RIGHT! ?

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