Phys 5701 8 Oct 2020

Propagators and Path Integrals

$$|\alpha, t_0; t\rangle = \exp\left[-\frac{i}{\hbar} H (t-t_0)\right] |\alpha, t_0\rangle$$

$$= \sum_{a'} \exp\left[-\frac{i}{\hbar} H (t-t_0)\right] |a'\rangle \langle a'|\alpha, t_0\rangle$$

$$= \sum_{a'} |a'\rangle \underbrace{\langle a'|\alpha, t_0\rangle} \exp\left[-\frac{i}{\hbar} E_{a'} (t-t_0)\right]$$

$$\langle \vec{x}''|\alpha, t_0; t\rangle$$

$$= \sum_{a'} \langle \vec{x}''|a'\rangle \langle a'|\alpha, t_0\rangle \, e^{-iE_{a'}(t-t_0)/\hbar}$$

$$\langle a'|\alpha, t_0\rangle = \int d^3x' \langle a'|\vec{x}'\rangle \langle \vec{x}'|\alpha, t_0\rangle$$

$$\to \psi(\vec{x}'', t) = \int d^3x' \, K(\vec{x}'', t; \vec{x}', t_0) \, \psi(\vec{x}', t_0)$$

$$K(\vec{x}'', t; \vec{x}', t_0) = \sum_{a'} \langle \vec{x}'' | a' \rangle \langle a' | \vec{x}' \rangle \, e^{-iE_{a'}(t-t_0)/\hbar}$$

$$= \text{"Propagator"}$$

Observations

1) K predicts the future! "Causal"

2) $K(\vec{x}'', t)$ with $\vec{x}', t_0$ fixed satisfies the wave equation!

3) $\lim_{t \to t_0} K(\vec{x}'', t; \vec{x}', t_0)$

$$= \sum_{a'} \langle \vec{x}'' | a' \rangle \langle a' | \vec{x}' \rangle = \delta^{(3)}(\vec{x}'' - \vec{x}')$$

"Green's Function"

K as a "Transition Amplitude"

$$K(\vec{x}'',t;\vec{x}',t_0) = \sum_{a'} \langle \vec{x}''|a'\rangle\langle a'|\vec{x}'\rangle\, e^{-iE_{a'}(t-t_0)/\hbar}$$

$$= \sum_{a'} \langle \vec{x}''|\, e^{-iH(t-t_0)/\hbar}\, |a'\rangle\langle a'|\vec{x}'\rangle$$

$$= \langle \vec{x}''|\,\mathcal{U}(t,t_0)\,|\vec{x}'\rangle$$

$$= \langle \vec{x}'',t\,|\,\vec{x}',t_0\rangle$$

Path Integral Formulation

q_I' = set of initial coordinates

q_F' = final

$$\boxed{K = \langle q_F'|\, e^{-iHt/\hbar}\, |q_I'\rangle}$$

Insert N "screens"

$$= \langle q_F'|\, e^{-iH\delta t/\hbar}_{(1)}\, e^{-iH\delta t/\hbar}_{(2)} \cdots e^{-iH\delta t/\hbar}_{(N)}\, |q_I'\rangle$$

with $\delta t = t/N$

Insert $N-1$ complete sets of states

i.e. $\mathbb{1} = \int dq_j' \, |q_j'\rangle \langle q_j'| \quad j = 1, 2, \ldots, N-1$

$$K = \left[\prod_{j=1}^{N-1} \int dq_j \right] \langle q_F' | e^{-iH\delta t/\hbar} | q_{N-1}' \rangle \langle q_{N-1}' | e^{-iH\delta t/\hbar} | q_{N-2}' \rangle \cdots \langle q_1' | e^{-iH\delta t/\hbar} | q_I' \rangle$$

Use $H = \frac{1}{2m} p^2 \; \left(+ V(q) \right)$

$$\boxed{\langle q_{j+1}' | e^{-i(p^2/2m)\delta t/\hbar} | q_j' \rangle}$$

$$= \int dp' \, \langle q_{j+1}' | e^{-i(p^2/2m)\delta t/\hbar} | p' \rangle \langle p' | q_j' \rangle$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dp' \, e^{-i(p'^2/2m)\delta t/\hbar + ip'(q_{j+1}' - q_j')/\hbar}$$

"Gaussian Integral"

Complete the square

$$-i\frac{p'^2}{2m}\frac{\delta t}{\hbar}+ip'(q'_{j+1}-q'_j)/\hbar$$

$$=-i\frac{\delta t}{2m\hbar}\Big[\underbrace{p'^2-\frac{2m}{\delta t}p'(q'_{j+1}-q'_j)+\frac{m^2}{\delta t^2}(q'_{j+1}-q'_j)^2}_{=\left[p'-\frac{m}{\delta t}(q'_{j+1}-q'_j)\right]^2}\Big]$$

$$+i\frac{\delta t}{2m\hbar}\frac{m^2}{\delta t^2}(q'_{j+1}-q'_j)^2$$

$$\langle q'_{j+1}|e^{-i(p^2/2m)\delta t/\hbar}|q_j\rangle$$

$$=\left(\frac{-i2m\hbar}{\delta t}\right)^{1/2}e^{i\frac{\delta t}{\hbar}\frac{m}{2}\left(\frac{q'_{j+1}-q'_j}{\delta t}\right)^2}$$

$$K=\prod_{j=1}^{N-1}\int dq'_j\left(\frac{-i2m\hbar}{\delta t}\right)^{N/2}e^{i\frac{\delta t}{\hbar}\frac{m}{2}\sum_j\left(\frac{q'_{j+1}-q'_j}{\delta t}\right)^2}$$

$$\xrightarrow[N\to\infty]{\delta t\to 0}\int Dq(t)\,e^{i\int_0^t dt\,\frac{1}{2}m\dot{q}^2}$$

Add $V(q)$ to H $\rightarrow$ $\int Dq(t)\,e^{\frac{i}{\hbar}\int_0^t dt\left[\frac{1}{2}m\dot{q}^2-V(q)\right]}$

$$\Rightarrow K = \int Dq(t)\, e^{iS[q(t)]/\hbar}$$

$$S[q(t)] = \int dt\, \underbrace{\left[\frac{1}{2} m \dot{q}^2 - V(q) \right]}_{L(q,\dot{q})}$$

- Hard to calculate! But elegant!
- All paths contribute! But...

  if $S \gg \hbar \Rightarrow$ Only min S contributes!

  $\Rightarrow$ Classical Physics

  if $S \sim \hbar \Rightarrow$ Quantum mechanics!

- Field theory! $L = \int d^3x\, \mathcal{L}$