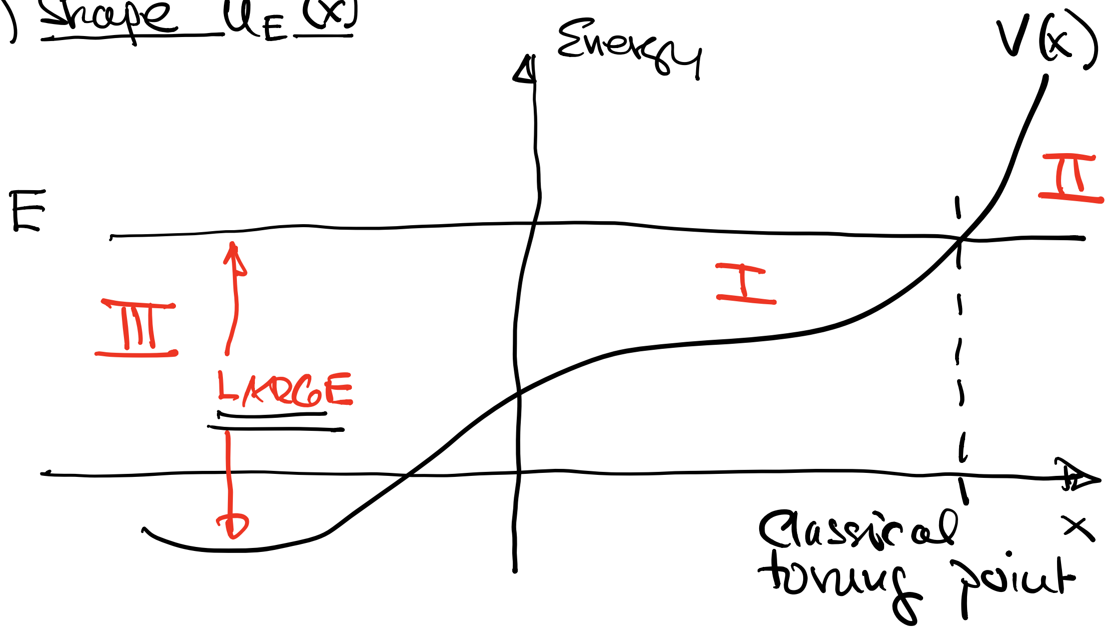
Phys 5701 6 Oct 2020

1D Potentials: General Observations

A) Shape $u_E(x)$



I: $\frac{d^2 u_E}{dx^2} = -\underbrace{\frac{2m}{\hbar^2}\left[V(x) - E\right]}_{<0} u_E(x)$

$\Rightarrow$ Oscillations "Quantum Mech"

II: $d^2 u_E / dx^2 = (+)\, u_E(x) \Rightarrow$ Exponential

III: $d^2 u_E / dx^2$ "very negative" $\times\, u_E(x)$

$\Rightarrow$ Rapid oscillations ("semi-classical")

(B) Symmetric Potentials

Suppose $\boxed{V(-x) = V(x)}$ e.g. SHO!

$$-\frac{\hbar^2}{2m}\frac{d^2u_E}{dx^2} + V(x)u_E(x) = Eu_E(x)$$

x to −x

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u_E(-x) + V(+x)u_E(-x) = Eu_E(-x)$$

$\Rightarrow$ $u_E(-x)$ solves same equation as $u_E(x)$!!

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i.e. $u_E(-x) = e^{i\alpha}u_E(x)$

$$u_E(x) = u_E(-(-x)) = e^{2i\alpha}u_E(x)$$

$$\Rightarrow e^{2i\alpha} = 1 \qquad \alpha = 0, \pi$$

i.e. $u_E(-x) = u_E(x)$ "Positive Parity"

or $u_E(-x) = -u_E(x)$ "Negative Parity"

Implications for u_E $(x=0)$

Negative: $u_E(0) = u_E(-0) = -u_E(0)$

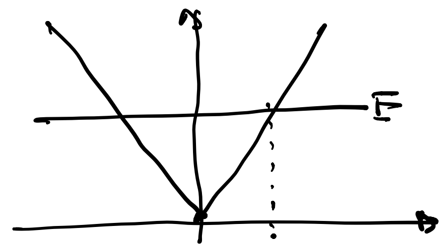
$$\Leftrightarrow \quad u_E(0) = 0$$

Positive: $u_E'(0) = \lim_{\varepsilon \to 0} \dfrac{u_E(\varepsilon) - u_E(-\varepsilon)}{2\varepsilon}$

$$u_E'(0) = 0$$

The Linear Potential: $V(x) = k|x| \quad k>0$

$$-\frac{\hbar^2}{2m}\frac{d^2 u_E}{dx^2} + k|x|\,u_E(x) = E\,u_E(x)$$



NOTE: $V(-x) = V(x)$ symmetric!

Want "scales":

$$x_0 = (\hbar^2/mk)^{1/3} \quad * \text{ Distance}$$

$$\text{Energy} \quad kx_0 = (\hbar^2 k^2/m)^{1/3}$$

$[x_0] = L \qquad \hbar, m, k \Rightarrow x_0 = \hbar^x m^y k^z$

$[\hbar] = L^2 M T^{-1} \qquad [m] = M$

$[k] = [\text{Energy}]/L = M L T^{-2}$

$\Rightarrow L = L^{2x} M^x T^{-x} M^y M^z L^z T^{-2z}$

$x + y + z = 0 \qquad 2x + z = 1 \qquad x + 2z = 0$

$4x - x = 2 \Rightarrow x = 2/3$

$y = -2/3 + 1/3 = -1/3$

$\Rightarrow z = -1/3$

$x_0 = \hbar^{2/3} m^{-1/3} z^{-1/3} = (\hbar^2/mk)^{1/3}$ *

Define $y \equiv x/x_0 \quad \varepsilon = E/kx_0$

$$-\frac{\hbar^2}{2m}\frac{1}{x_0^2}\frac{d^2u}{dy^2} + kx_0 |y| \overset{u(y)}{\cdot} = kx_0\, \varepsilon\, u(y) \qquad \div kx_0$$

$$-\frac{\cancel{\hbar^2}}{\cancel{2m}}\frac{1}{\cancel{k}}\frac{\cancel{mk}}{\cancel{\hbar^2}}\frac{d^2u}{dy^2} + |y|u(y) = \varepsilon u(y)$$

Also $|y|$ is symmetric! $|-y| = |y|$

$\Rightarrow$ Only consider $y \geq 0$

$$-\frac{1}{2}\frac{d^2u}{dy^2} + y\,u(y) = \varepsilon\,u(y)$$

$$\Rightarrow \quad \frac{d^2u}{dy^2} - 2(y-\varepsilon)\,u(y) = 0$$

"classical turning point" $E = V(a) = ka$

i.e. $y = x/x_0 = a/x_0 = E/kx_0 = \varepsilon$

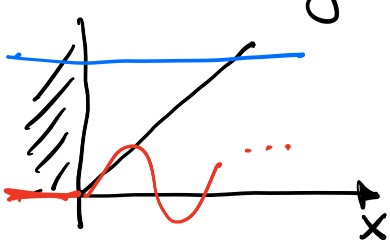
Write $z = 2^{1/3}(y-\varepsilon)$

$$\Rightarrow \quad \boxed{\frac{d^2u}{dz^2} - z\,u(z) = 0}$$

Airy Equation

Application: Bouncing Balls

$V(x) =$ "mgh" $= mgx$ i.e. $k = mg$



Need $u_E(x=0) = 0$

$\Rightarrow$ Negative Parity Solutions!

Observable?? "Height" $x_0 = (\hbar^2/mk)^{1/3}$

$$= (\hbar^2/m^2 g)^{1/3}$$

$m = m_{\text{NEUTRON}} \Rightarrow x_0 = 7.4\,\mu m$