Phys 5710 5 Nov 2020

Review: Orbital Angular Momentum

$\vec{L} = \vec{x} \times \vec{p}$ operator in position space

$L_i = \varepsilon_{ijk} x_j p_k$ for $i,j,k = x,y,z$

$\Rightarrow [L_i, L_j] = i\hbar\, \varepsilon_{ijk} L_k$ "Angular Momentum"

For $D_z(\delta\phi) = 1 - \frac{i}{\hbar} L_z \, \delta\phi$ $\leftarrow$

$\Rightarrow \langle x', y', z' | D_z(\delta\phi) | \alpha\rangle$

$= \langle x' + y'\delta\phi, y' - x'\delta\phi, z' | \alpha\rangle$

$x' = r\cos\phi \sin\theta$

$y' = r\sin\phi \sin\theta$

$z' = r\cos\theta$

Obvious rotation!

$\Rightarrow \langle r, \theta, \phi | D_z(\delta\phi) | \alpha\rangle$

$= \langle r, \theta, \phi - \delta\phi | \alpha\rangle$

$\Rightarrow$ Find position representation of L_z and $\vec{L}^2$ on $|\alpha\rangle$

$$\langle \vec{x}' | L_z | \alpha \rangle = -i\hbar \frac{\partial}{\partial \phi} \langle \vec{x}' | \alpha \rangle \longleftarrow *$$

Recall: $\langle \vec{x}' | p_x | \alpha \rangle = -i\hbar \frac{\partial}{\partial x} \langle \vec{x}' | \alpha \rangle$

also: $\langle \vec{x}' | L_{\pm} | \alpha \rangle \qquad L_{\pm} = L_x \pm i L_y$

$$= -i\hbar\, e^{\pm i\phi} \left[\pm i \frac{\partial}{\partial \theta} - \cot\theta \frac{\partial}{\partial \phi} \right] \langle \vec{x}' | \alpha \rangle$$

and

$$\langle \vec{x}' | \vec{L}^2 | \alpha \rangle$$

$$= -\hbar^2 \left[\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) \right] \langle \vec{x}' | \alpha \rangle$$

Eigenfunctions of $\vec{L}^2$ and L_z

$[\vec{L}^2, L_z] = 0 \Rightarrow$ Eigenstates $|lm\rangle$

w/ $\vec{L}^2 |lm\rangle = l(l+1)\hbar^2 |lm\rangle$

$L_z |lm\rangle = m\hbar |lm\rangle$

$l = 0, 1/2, 1, 3/2, \ldots$ (?)

and $m = -l, -l+1, \ldots, l-1, l$ ($2l+1$ values)

Our goal: Find $\langle \vec{x}' | lm\rangle = \langle \hat{n}' | lm\rangle$

$$\equiv Y_l^m(\theta, \phi)$$

"Spherical Harmonics"

First: $|\alpha\rangle = |lm\rangle$

i.e. $L_z|\alpha\rangle = m\hbar |lm\rangle$

$\vec{L}^2 |\alpha\rangle = l(l+1)\hbar^2 |lm\rangle$

Next: $\langle \hat{n} | L_z |\alpha\rangle = m\hbar \langle \hat{n} | lm\rangle$ ✓

$$= -i\hbar \frac{\partial}{\partial \phi} \langle \hat{n} | lm\rangle \quad ✓$$

Put $\langle \hat{n} | \ell m \rangle = f(\theta)\, g(\phi)$

$$\& \quad -i\hbar \frac{\partial}{\partial\phi}\left[f(\theta)\, g(\phi)\right] = m\hbar\, f(\theta)\, g(\phi)$$

$$\text{i.e.} \quad \frac{dg}{d\phi} = im\, g(\phi) \Rightarrow g(\phi) = e^{im\phi}$$

<u>Now the hard part!</u>

$$\langle \hat{n}' | \vec{L}^2 | \ell m \rangle = \ell(\ell+1)\hbar^2 \langle \hat{n} | \ell m \rangle$$

$$= \ell(\ell+1)\cancel{\hbar^2}\, \cancel{e^{im\phi}}\, f(\theta)$$

$$= -\hbar^2 \left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right)\right] e^{im\phi} f(\theta)$$

$$= -\cancel{\hbar^2}\left[\frac{1}{\sin^2\theta}(-m^2)\, f(\theta) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{df}{d\theta}\right)\right] \cancel{e^{im\phi}}$$

$$+\frac{m^2}{\sin^2\theta} f(\theta) - \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{df}{d\theta}\right) = \ell(\ell+1)\, f(\theta)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{df}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] f(\theta) = 0$$

$x = \cos\theta \qquad -1 \le x \le 1$

$dx = -\sin\theta \, d\theta \qquad P(x) \equiv f(\theta)$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\right) = \frac{d}{dx}\left(\sin^2\theta \frac{d}{dx}\right)$$

$$\Leftrightarrow \frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P(x) = 0$$

or

$$\boxed{(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P(x) = 0}$$

"Associated Legendre Equation"
Arfken & Weber 4e (12.72)

Solutions $P_\ell^m(x)$ "Assoc. Legendre Functions"

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$ Legendre Polynomials

$$\Rightarrow \underline{\underline{P_\ell^m(x)}} = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x) \qquad m \ge 0$$

and $$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)$$

Spherical Harmonics

$$\langle \vec{x}' | \ell m \rangle \equiv Y_\ell^m(\theta,\phi)$$

$$= (-1)^m \left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right]^{1/2} P_\ell^m(\cos\theta)\, e^{im\phi}$$

with $\displaystyle\int Y_{\ell'}^{m'*}(\theta,\phi)\, Y_\ell^m(\theta,\phi) \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega} = 1$

where $\displaystyle\int_{-1}^{1} P_\ell^m(x)\, P_\ell^m(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}$

Does ℓ need to be an integer?

1) Suppose $m = 1/2 \cdot$ integer

$\Rightarrow Y_\ell^m(\theta, \phi+2\pi) = -Y_\ell^m(\theta,\phi)$

2) Sturm-Liouville Theory

integer ℓ form a complete set.

$\Rightarrow$ $1/2$ integer ℓ's not independent.

3) Recall $\langle \hat{n}' | L_\pm | \alpha \rangle = -i\hbar\, \underbrace{e^{\pm i\phi}\left[\pm i \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi}\right]}_{} \langle \hat{n}' | \alpha \rangle$

Put $|\alpha\rangle = |\ell, m=\ell\rangle \Rightarrow \underbrace{L_+ |\ell\ell\rangle}_{} = 0$

$\Leftrightarrow \left[i \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi}\right] Y_\ell^\ell(\theta,\phi) = 0$

Put $Y_\ell^\ell(\theta,\phi) = e^{i\ell\phi} f(\theta)$

$\Leftrightarrow \cancel{i} \frac{df}{d\theta} - \cancel{i}\ell \cot\theta \; f(\theta) = 0$

$\Leftrightarrow f(\theta) = \text{const} \times \sin^\ell(\theta)$ (Easy!!)

i.e. $\boxed{Y_\ell^\ell(\theta,\phi) = C_\ell\, e^{i\ell\phi} \sin^\ell(\theta)}$

Now put $\ell = 1/2$ and act with L_-:

$Y_{1/2}^{1/2}(\theta,\phi) = C_{1/2}\, e^{i\phi/2} \sin^{1/2}(\theta)$

$\langle \hat{n}' | L_- | \tfrac{1}{2}\tfrac{1}{2}\rangle = -i\hbar\, e^{-i\phi}\left[-i\frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi}\right] Y_{1/2}^{1/2}(\theta,\phi)$

$= -i\hbar\, e^{-i\phi/2} C_{1/2} \Big[-i \frac{1}{2} \sin^{-1/2}\theta \;\cos\theta$

$- i \frac{1}{2} \cot\theta \;\sin^{1/2}\theta \Big]$

$\Rightarrow$ $Y_{1/2}^{-1/2}(\theta,\phi) = \text{const} \times \boxed{\sin^{1/2}\theta \; \cot\theta \; e^{-i\phi/2}}$

But I can also get here by...

$$L_- |1/2, -1/2\rangle = 0$$

$$\Rightarrow -i\hbar e^{-i\phi}\left[+i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right] Y_{1/2}^{-1/2} = 0$$

Put $Y_{1/2}^{-1/2} = e^{-i\phi/2} f(\theta)$

$$\Rightarrow +i\frac{df}{d\theta} + \frac{i}{2}\cot\theta \, f(\theta) = 0$$

i.e. $\dfrac{df}{d\theta} + \dfrac{1}{2}\cot\theta \, f(\theta) = 0$

$$\Rightarrow Y_{1/2}^{-1/2}(\theta,\phi) = \boxed{e^{-i\phi/2}\sin^{1/2}(\theta)}$$

$\times$ constant.

INCONSISTENT!

$\Rightarrow$ l must be $0, 1, 2, \ldots$