Phys 5701 3 Sep 2020

Last Class $A|a'\rangle = a'|a'\rangle$

"Complete" $|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle \quad c_{a'} = \langle a'|\alpha\rangle$

$$= \sum_{a'} |a'\rangle\langle a'|\alpha\rangle = 1|\alpha\rangle$$

$$\langle\alpha|\alpha\rangle = 1 \Rightarrow \sum_{a'} |\langle a'|\alpha\rangle|^2 = 1$$

Experiment $\langle A\rangle = \langle\alpha|A|\alpha\rangle$ theory

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"Magic" $|\alpha\rangle \xrightarrow{\text{Measure } A} |a'\rangle$

w/ Probability $|\langle a'|\alpha\rangle|^2$

Example: $S_z = \frac{\hbar}{2}\left[|+\rangle\langle+| - |-\rangle\langle-|\right]$

$$\Leftrightarrow S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

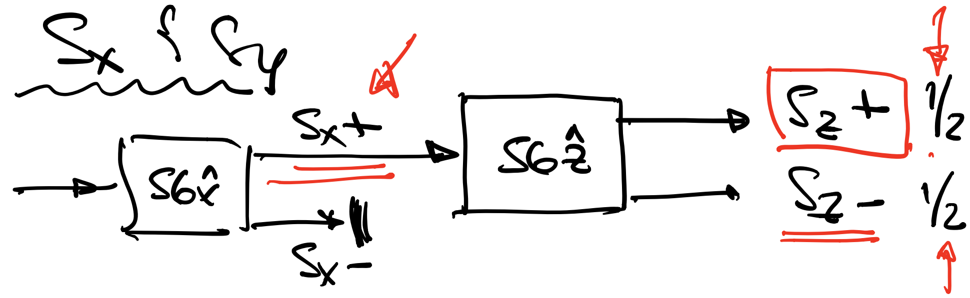
also same for S_x, S_y

Also: $S_+ \equiv \hbar|+\rangle\langle-|$

$S_- \equiv \hbar|-\rangle\langle+|$

Building S_x & S_y

Recall:



$\rightarrow$ i.e. $|\langle + | S_x; +\rangle|^2 = \frac{1}{2}$

$\Rightarrow$ $|\langle + | S_x; +\rangle| = \frac{1}{\sqrt{2}}$ $\quad |\langle - | S_x; +\rangle| = \frac{1}{\sqrt{2}}$

$$\boxed{\begin{aligned} \Rightarrow |S_x;+\rangle &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}} e^{i\delta_1}|-\rangle \\ |S_x;-\rangle &= \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}} e^{i\delta_1}|-\rangle \end{aligned}}$$

$$0 = \langle S_x;+|S_x;-\rangle = \frac{1}{2}\left[1+0+0+e^{i(\delta_0-\delta_1)}\right]$$

i.e. $\delta_0 - \delta_1 = \pi$

$$\Rightarrow e^{i\delta_0} = e^{i\pi} e^{i\delta_1} = -e^{-i\delta_0}$$

also $$\boxed{|S_y;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}} e^{i\delta_2}|-\rangle}$$

But $\left| \langle S_y; \pm | S_x; + \rangle \right| = \frac{1}{\sqrt{2}}$

$$\left| \left[\frac{1}{\sqrt{2}} \langle +| \pm \frac{1}{\sqrt{2}} e^{-i\delta_2} \langle -| \right] \left[\frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\delta_1} |-\rangle \right] \right|$$

$$= \left| \frac{1}{2} + 0 \pm 0 \pm \frac{1}{2} e^{i(\delta_1 - \delta_2)} \right| = \frac{\sqrt{2}}{2}$$

$$\sqrt{\left(1 \pm e^{-i(\delta_1 - \delta_2)}\right)\left(1 \pm e^{i(\delta_1 - \delta_2)}\right)} = \sqrt{2}$$

$$\sqrt{1 + 1 \pm 2 \underbrace{\cos(\delta_1 - \delta_2)}_{=0}} = \sqrt{2}$$

$$\Leftrightarrow \delta_1 - \delta_2 = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

<u>CONVENTION</u>: $\delta_1 = 0$

RIGHT HAND COORDINATE SYSTEM

$$\Rightarrow \delta_2 = +\pi/2$$

$$|S_x; \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$$

$$|S_y; \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$$

<u>GREAT!</u>

Recall $S_z = \frac{\hbar}{2}\left[\,|+\rangle\langle+| - |-\rangle\langle-|\,\right]$

$$S_x = \frac{\hbar}{2}\left[\,|S_x;+\rangle\langle S_x;+| - |S_x;-\rangle\langle S_x;-|\,\right]$$

$$= \frac{\hbar}{2}\Big\{ \frac{1}{2}\left[|+\rangle + |-\rangle\right]\left[\langle+| + \langle-|\right] - \frac{1}{2}\left[|+\rangle - |-\rangle\right]\left[\langle+| - \langle-|\right] \Big\}$$

$$= \frac{\hbar}{2}\Big\{ \frac{1}{\cancel{2}}\left[\,|+\rangle\langle+| + |+\rangle\langle-| + |-\rangle\langle+| + |-\rangle\langle-|\,\right] - \frac{1}{\cancel{2}}\left[\,|+\rangle\langle+| - |+\rangle\langle-| - |-\rangle\langle+| + |-\rangle\langle-|\,\right] \Big\}$$

↳ $S_x = \frac{\hbar}{2}\left[\, |+\rangle\langle -| + |-\rangle\langle +| \,\right]$

$S_y = \frac{\hbar}{2}\left[\, -i\,|+\rangle\langle -| + i\,|-\rangle\langle +| \,\right]$

NOTE: $S_\pm = S_x \pm i S_y$

Definitions: $[A, B] \equiv AB - BA$

$\{A, B\} \equiv AB + BA$

PROVE THESE: $\{S_i, S_j\} = \frac{1}{2}\hbar^2 \delta_{ij}$

$[S_i, S_j] = i\, \varepsilon_{ijk}\, \hbar\, S_k$ sum

e.g. $[S_x, S_y] = i\hbar S_z$

$\vec{S}^2 \equiv S_x^2 + S_y^2 + S_z^2 = \frac{3}{4}\hbar^2 \mathbb{1}$

↳ $[\vec{S}^2, S_i] = 0$ $\quad i = x, y, z$

Compatible Observables

A, B compatible iff $[A,B]=0$

Theorem: If A, B compatible then B is diagonal in A basis.

i.e. $\langle a''|B|a'\rangle = \langle a'|B|a'\rangle \delta_{a'a''}$

Proof: $\langle a''|[A,B]|a'\rangle = 0$

$= \langle a''|(AB-BA)|a'\rangle$

$= (a''-a')\langle a''|B|a'\rangle$

Different states $\Rightarrow \langle a''|B|a'\rangle = 0$

NON DEGENERATE.

$$B = \mathbb{1}\, B\, \mathbb{1} = \sum_{a'} \sum_{a''} |a''\rangle\langle a''|B|a'\rangle\langle a'|$$

$$= \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|$$

$$B|a'\rangle = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|a'\rangle$$

$$= \underbrace{\langle a'|B|a'\rangle}_{\equiv b'}\, |a'\rangle$$

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we write $A\,|a',b'\rangle = a'\,|a',b'\rangle$

$B\,|a',b'\rangle = b'\,|a',b'\rangle$

The Uncertainty Relation

Define operator $\Delta A = A - \langle A \rangle$ (1)

Consider $\langle (\Delta A)^2 \rangle$

$$= \langle (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \rangle$$

$$= \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2$$

$$= \langle A^2 \rangle - \langle A \rangle^2 \quad \text{"Variance"}$$

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e.g. For the state $|\alpha\rangle = |+\rangle$

$$\langle S_z \rangle = \langle + | S_z | + \rangle = \frac{\hbar}{2}$$

$$\langle S_z^2 \rangle = \left(\frac{\hbar}{2}\right)^2 \Rightarrow \langle (\Delta S_z)^2 \rangle = 0$$

$$\langle S_x \rangle = \langle + | \frac{\hbar}{2} [|+\rangle\langle -| + |-\rangle\langle +|] |+\rangle = 0$$

$$\langle S_x^2 \rangle = \frac{\hbar^2}{4} \Rightarrow \langle (\Delta S_x)^2 \rangle = \frac{\hbar^2}{4}$$

Now: $\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle$

$$\geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2$$