Phys 5701 3 Nov 2020

Review: Angular Momentum

Infinitesimal rotations about axis $\hat{n}$

$$D(\hat{n}, d\phi) = 1 - \frac{i}{\hbar} \vec{J}\cdot\hat{n}\, d\phi$$

$\vec{J} = J_x \hat{x} + J_y \hat{y} + J_z \hat{z}$ "Generator of rotations"

Rotations do not commute!

$$\Rightarrow \boxed{\begin{array}{l} [J_x, J_y] = i\hbar J_z \\ [J_z, J_x] = i\hbar J_y \; ; \; [J_y, J_z] = i\hbar J_x \end{array}}$$

i.e. $[J_i, J_j] = \sum_k i\varepsilon_{ijk} J_k$

Eigenvalues & Eigenstates $|jm\rangle$

$\vec{J}^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle$ $\quad \vec{J}^2 = J_x^2 + J_y^2 + J_z^2$

$J_z |jm\rangle = m\hbar |jm\rangle$

$\boxed{\text{w/ } j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots}$

$m = -j, -j+1, \ldots, j-1, j$ $\quad$ ($2j+1$ values)

Today: Orbital Angular Momentum

$$\vec{L} = \vec{x} \times \vec{p}$$ Operator in position space!

$$\left.\begin{aligned} L_x &= y p_z - z p_y \\ L_y &= z p_x - x p_z \\ L_z &= x p_y - y p_x \end{aligned}\right\} \quad L_i = \varepsilon_{ijk} x_j p_k \qquad i,j,k = x,y,z$$

$$\boxed{[L_x, L_y]} = (y p_z - z p_y)(z p_x - x p_z) - (z p_x - x p_z)(y p_z - z p_y)$$

$$= y p_z z p_x - y p_z x p_z - z p_y z p_x + z p_y x p_z$$
$$- z p_x y p_z + x p_z y p_z + z p_x z p_y - x p_z z p_y$$
$$=0 \qquad\qquad =0$$

$$= y p_x (p_z z - z p_z) + x p_y (z p_z - p_z z)$$

$$= y p_x (-i\hbar) + x p_y (+i\hbar)$$

$$= i\hbar (x p_y - y p_x) = \boxed{i\hbar L_z}$$ Right!

The Cool Way!

$$[L_i, L_j] = [\varepsilon_{ikl} x_k p_l , \varepsilon_{jmn} x_m p_n]$$

$$= \varepsilon_{ikl}\varepsilon_{jmn}\left(\underline{x_k p_l x_m p_n - x_m p_n x_k p_l}\right)$$

$$\underline{x_k p_l x_m p_n} = x_k (x_m p_l - i\hbar \delta_{ml}) p_n$$

$$= x_k x_m p_l p_n - i\hbar \delta_{ml} x_k p_n$$

$$\underline{x_m p_n} x_k p_l = x_m (x_k p_n - i\hbar \delta_{kn}) p_l$$

$$= x_m x_k p_n p_l - i\hbar \delta_{kn} x_m p_l$$

$$[L_i, L_j] = \varepsilon_{ikl}\varepsilon_{jmn} (-i\hbar) \delta_{ml} x_k p_n$$

$$- \varepsilon_{ikl}\varepsilon_{jmn} (-i\hbar) \delta_{kn} x_m p_l$$

$$= \varepsilon_{ikl}\varepsilon_{jln} (-i\hbar) x_k p_n$$

$$- \varepsilon_{ikl}\varepsilon_{jmk} (-i\hbar) x_m p_l$$

$$= i\hbar\, \varepsilon_{li\cancel{k}}{}^{m} \varepsilon_{ljn} x_{\cancel{k}}{}^{m} p_n$$

$$- i\hbar\, \varepsilon_{ki\cancel{l}}{}^{n} \varepsilon_{kjm} x_m p_{\cancel{l}}{}^{n}$$

$$= i\hbar \left(\varepsilon_{\ell i m}\varepsilon_{\ell j n} - \varepsilon_{k i n}\varepsilon_{k j m}\right) x_m p_n$$

<u>THEOREM</u>: $\varepsilon_{imn}\varepsilon_{ipq} = \delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}$!!

$$\Rightarrow \varepsilon_{\ell i m}\varepsilon_{\ell j n} = \delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}$$

$$\varepsilon_{k i n}\varepsilon_{k j m} = \delta_{ij}\delta_{mn} - \delta_{im}\delta_{nj}$$

$$\Rightarrow [L_i, L_j] = i\hbar \left(\delta_{im}\delta_{nj} - \delta_{in}\delta_{mj}\right) x_m p_n$$

$$= i\hbar\, \varepsilon_{kij}\, \underbrace{\varepsilon_{kmn} x_m p_n}_{L_k}$$

$$= i\hbar\, \varepsilon_{kij} L_k$$

$$= i\hbar\, \varepsilon_{ijk} L_k$$

$\Rightarrow$ Angular Momentum Commutation Relations!

$\vec{L}$ as the rotation operator generator

$$D_z(\delta\phi)\,|\vec{x}'\rangle = \left[1-\frac{i}{\hbar}L_z\,\delta\phi\right]|x',y',z'\rangle$$

$$L_z = xp_y - yp_x = p_y x - p_x y$$

$$D_z(\delta\phi)\,|\vec{x}'\rangle = \left[1-\frac{i}{\hbar}p_y\,(x'\delta\phi) + \frac{i}{\hbar}p_x\,(y'\delta\phi)\right]|\vec{x}'\rangle$$

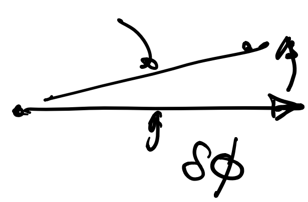
$$\mathcal{J}(d\vec{x}') = 1-\frac{i}{\hbar}\vec{p}\cdot d\vec{x}'$$

$$\Rightarrow D_z(\delta\phi) = \mathcal{J}_x(-y'\delta\phi)\,\mathcal{J}_y(x'\delta\phi)$$

i.e. $\langle\vec{x}'|D_z(\delta\phi)|\alpha\rangle = \langle x',y',z'|D_z(\delta\phi)|\alpha\rangle$

$$= \langle x'+y'\delta\phi,\; y'-x'\delta\phi,\; z'|\alpha\rangle$$

e.g.



Spherical Coordinates

$$x' = r\cos\phi\sin\theta \quad *$$
$$y' = r\sin\phi\sin\theta \qquad \text{and do} \quad \phi \to \phi - \delta\phi$$
$$z' = r\cos\theta$$

$$\Rightarrow \cos\phi \to \cos(\phi - \delta\phi)$$
$$= \cos\phi\cos\delta\phi + \sin\phi\sin\delta\phi$$
$$= \cos\phi + \delta\phi\,\underline{\sin\phi}$$
$$\sin\phi \to \sin\phi - \delta\phi\cos\phi$$

$$\Rightarrow x' \to x' + \delta\phi\,\underline{r\sin\phi\sin\theta} = x' + y'\delta\phi$$

also $y' \to y' - x'\delta\phi$

$$\langle r,\theta,\phi|\mathcal{D}_z(\delta\phi)|\alpha\rangle = \langle r,\theta,\phi-\delta\phi|\alpha\rangle$$

$$= \langle r,\theta,\phi|\left[\underline{1} - \frac{i}{\hbar}L_z\,\delta\phi\right]|\alpha\rangle$$

$$\underline{\langle r,\theta,\phi|\alpha\rangle} - \delta\phi\frac{\partial}{\partial\phi}\langle r,\theta,\phi|\alpha\rangle$$

$$-\frac{i}{\hbar}\langle r,\theta,\phi|L_z|\alpha\rangle = \frac{\partial}{\partial\phi}\langle r,\theta,\phi|\alpha\rangle$$

i.e. $\boxed{\langle \vec{x}' | L_z | \alpha \rangle = -i\hbar \frac{\partial}{\partial \phi} \langle \vec{x}' | \alpha \rangle}$ $\ast$

Compare to $\langle \vec{x}' | \vec{p} | \alpha \rangle = -i\hbar \vec{\nabla}' \langle \vec{x}' | \alpha \rangle$

"Similarly"

$$\boxed{\begin{aligned} \langle \vec{x}' | L_x | \alpha \rangle &= -i\hbar \left[-\sin\phi \frac{\partial}{\partial \theta} - \cot\theta \cos\phi \frac{\partial}{\partial \phi} \right] \langle \vec{x}' | \alpha \rangle \\ \langle \vec{x}' | L_y | \alpha \rangle &= -i\hbar \left[\cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi} \right] \langle \vec{x}' | \alpha \rangle \end{aligned}}$$

For $L_\pm \equiv L_x \pm i L_y$

$$\boxed{\langle \vec{x}' | L_\pm | \alpha \rangle = -i\hbar e^{\pm i\phi} \left[\pm i \frac{\partial}{\partial \theta} - \cot\theta \frac{\partial}{\partial \phi} \right] \langle \vec{x}' | \alpha \rangle}$$

will find $\langle \vec{x}' | \ell m \rangle$ "Eigenfunctions"

e.g. $\langle r, \theta, \phi | L_z | \ell m \rangle = m\hbar \langle r, \theta, \phi | \ell m \rangle$

$$= -i\hbar \frac{\partial}{\partial \phi} \langle r, \theta, \phi | \ell m \rangle$$

What about $\vec{L}^2$?? ...

$$\vec{L}^2 = \varepsilon_{ijk} x_i p_j \, \varepsilon_{lmk} x_l p_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \, x_i p_j x_l p_m$$

etc...

$$\Rightarrow \vec{L}^2 = \vec{x}^2 \vec{p}^2 - (\vec{x}\cdot\vec{p})^2 + i\hbar \, \vec{x}\cdot\vec{p}$$

$$\langle \vec{x}' | \vec{x}^2 \vec{p}^2 | \alpha \rangle = \vec{x}'^2 \langle \vec{x}' | \vec{p}^2 | \alpha \rangle$$

$$= r^2 \langle \vec{x}' | \vec{p}^2 | \alpha \rangle$$

$$\langle \vec{x}' | \vec{x}\cdot\vec{p} | \alpha \rangle = \vec{x}' \cdot \langle \vec{x}' | \vec{p} | \alpha \rangle$$

$$\langle \vec{x}' | (\vec{x}\cdot\vec{p})^2 | \alpha \rangle = -\hbar^2 \left[r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} \right] \langle \vec{x}' | \alpha \rangle$$

$$\Rightarrow \boxed{\langle \vec{x}' | \vec{L}^2 | \alpha \rangle = r^2 \underbrace{\langle \vec{x}' | \vec{p}^2 | \alpha \rangle}_{= -\hbar^2 \vec{\nabla}'^2} + \hbar^2 \left[r^2 \frac{\partial^2}{\partial r^2} + 2r \frac{\partial}{\partial r} \right] \langle \vec{x}' | \alpha \rangle}$$

Radial part of $\vec{\nabla}'^2$

1.e. $\langle \vec{x}' | \vec{L}^2 | \alpha \rangle$

$$= -\hbar^2 \left[\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) \right] \langle \vec{x}' | \alpha \rangle$$

$$\Rightarrow \langle \theta, \phi | l m \rangle \equiv Y_l^m(\theta, \phi)$$