Phys 5701 — 1 Sep 2022

Last Class

$|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|$ Dual Space

$c|\alpha\rangle \longleftrightarrow c^*\langle\alpha|$ Postulate

$X|\alpha\rangle \longleftrightarrow \langle\alpha|X^\dagger$ Definition

$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$ Postulate

$\langle\alpha|\alpha\rangle \geq 0$ Postulate

$(XY)^\dagger = Y^\dagger X^\dagger$ theorem

$(|\alpha\rangle\langle\beta|)^\dagger = |\beta\rangle\langle\alpha|$ theorem

$\langle\alpha|X|\beta\rangle = \langle\beta|X^\dagger|\alpha\rangle^*$ theorem

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Today: Representations

Thm: Eigenvalues of Hermitian Operators are real numbers and the eigenvectors are orthogonal.

Proof: $A|a'\rangle = a'|a'\rangle$

$\langle a''|A = a''^*\langle a''|$

$\Rightarrow \langle a''|A|a'\rangle = a'\langle a''|a'\rangle$

or $\langle a''|A|a'\rangle = a''^*\langle a''|a'\rangle$

$$\Rightarrow \boxed{(a' - a''^*)\langle a''|a'\rangle = 0}$$

"Eigenkets span the space"

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle \Longrightarrow \text{FIND } c_{a'} \text{? } \underline{\underline{\text{Yes}}}.$$

$$\langle a''|\alpha\rangle = \sum_{a'} c_{a'} \langle a''|a'\rangle = \sum_{a'} c_{a'} \delta_{a',a''} = c_{a''}$$

$$\Leftrightarrow \boxed{c_{a'} = \langle a'|\alpha\rangle}$$

$$\boxed{|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle = \underbrace{\left[\sum_{a'} |a'\rangle\langle a'|\right]}_{1} |\alpha\rangle}$$

Suppose $|\alpha\rangle$ is a normalized state vector

$$\langle\alpha|\alpha\rangle = 1$$

$$= \langle\alpha| \left[\sum_{a'} |a'\rangle\langle a'|\right] |\alpha\rangle$$

$$= \sum_{a'} \langle\alpha|a'\rangle\langle a'|\alpha\rangle$$

$$= \sum_{a'} \underbrace{|\langle a'|\alpha\rangle|^2}_{} = \sum_{a'} |c_{a'}|^2 = 1$$

PROBABILITIES!

Representations

$$X = 1\,X\,1 = \left[\sum_{a''} |a''\rangle\langle a''|\right] X \left[\sum_{a'} |a'\rangle\langle a'|\right]$$

$$= \sum_{a'}\sum_{a''} |a''\rangle \langle a''|X|a'\rangle \langle a'|$$

$$X \doteq \begin{bmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \cdots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

MATRIX

$$Z = XY$$

$$\langle a''|Z|a'\rangle = \sum_{a'''} \langle a''|X|a'''\rangle\langle a'''|Y|a'\rangle$$

Matrix Multiplication!

$$|\gamma\rangle = X|\alpha\rangle$$

$$\langle a'|\gamma\rangle = \langle a'|X|\alpha\rangle = \sum_{a''}\langle a'|X|a''\rangle\langle a''|\alpha\rangle$$

Column.

$$\langle\gamma| = \langle\alpha|X$$

$$\langle\gamma|a'\rangle = \sum_{a''}\langle\alpha|a''\rangle\langle a''|X|a'\rangle$$

Row of complex conjugates

Represent A in $|a'\rangle$ basis?

$$\text{i.e. } A \doteq \begin{bmatrix} \langle a^{(1)}|A|a^{(1)}\rangle & \langle a^{(1)}|A|a^{(2)}\rangle & \cdots \\ \vdots & \langle a^{(2)}|A|a^{(2)}\rangle & \cdots \\ \vdots & & \end{bmatrix}$$

$$= \begin{bmatrix} a^{(1)} & & & 0 \\ & a^{(2)} & & \\ & & a^{(3)} & \\ 0 & & & \ddots \end{bmatrix}$$ "Diagonal matrix"

<u>Example: Spin-1/2</u>

$A \rightarrow S_z$

Eigenstates $|S_z;\pm\rangle$ w/ eivs $\pm\frac{\hbar}{2}$

$$S_z|S_z;+\rangle = +\frac{\hbar}{2}|S_z;+\rangle$$

$$S_z|S_z;-\rangle = -\frac{\hbar}{2}|S_z;-\rangle$$

<u>NOTATION</u>

$$|S_z;\pm\rangle \longrightarrow |\pm\rangle$$

e.g. $\boxed{1 = |+\rangle\langle+| + |-\rangle\langle-|}$

$$1 \doteq \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

Recall $A = A\,\mathbb{1} = \sum_{a'} A|a'\rangle\langle a'| = \sum_{a'} a'|a'\rangle\langle a'|$

$\Rightarrow$ $$S_z = \left(+\frac{\hbar}{2}\right)|+\rangle\langle+|$$

$$+ \left(-\frac{\hbar}{2}\right)|-\rangle\langle-|$$

$$= \frac{\hbar}{2}\Big[\,|+\rangle\langle+| - |-\rangle\langle-|\,\Big]$$

<u>NOTE</u>: $S_z|+\rangle$

$$= \frac{\hbar}{2}\Big[\,|+\rangle\underbrace{\langle+|+\rangle}_{=1} - |-\rangle\underbrace{\langle-|+\rangle}_{=0}\,\Big]$$

$= \frac{\hbar}{2}|+\rangle$ Right answer!

$$S_z|-\rangle = -\frac{\hbar}{2}|-\rangle$$

Matrix Representation of S_z

$$S_z \doteq \begin{bmatrix} \langle +|S_z|+\rangle & \langle +|S_z|-\rangle \\ \langle -|S_z|+\rangle & \langle -|S_z|-\rangle \end{bmatrix}$$

$$= \frac{\hbar}{2}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\sigma_z \text{ "Pauli Matrix"}}$$

$$|+\rangle \doteq \begin{bmatrix} \langle +|+\rangle \\ \langle -|+\rangle \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|-\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\langle +| \doteq \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\langle -| \doteq \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Two New Operators

$$S_+ \equiv \hbar |+\rangle\langle -| \quad \text{i.e. } S_+|-\rangle = \hbar|+\rangle$$

$$S_- \equiv \hbar |-\rangle\langle +| \quad \text{i.e. } S_-|+\rangle = \hbar|-\rangle$$

$$S_+|+\rangle = 0 = S_-|-\rangle$$

$$S_+^\dagger = S_- \quad S_+ \doteq \hbar\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad S_- \doteq \hbar\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

WHERE ARE WE HEADED?

MEASUREMENT!

MEASURE A in $|\alpha\rangle$

$\longrightarrow$ Jumps into $|a'\rangle$

Which one? Don't know!

But Prob of landing in $|a'\rangle$

is $|\langle a'|\alpha\rangle|^2$.

WATCH THIS!

$$\langle A \rangle = \sum_{a'} a' \, |\langle a' | \alpha \rangle|^2$$

$$= \sum_{a'} a' \langle \alpha | a' \rangle \langle a' | \alpha \rangle$$

$$= \sum_{a'} \langle \alpha | A | a' \rangle \langle a' | \alpha \rangle$$

$$= \langle \alpha | A | \alpha \rangle$$ Expectation Value

$$a' \langle \alpha | a' \rangle = \langle \alpha | \left[a' | a' \rangle \right]$$

$$= \langle \alpha | A | a' \rangle$$