The Simple Pendulum

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This experiment is a straightforward measurement that will give you important experience in determining measurement uncertainties. You will construct a pendulum from a light string attached to a fixed point and from which hangs a heavy weight. (You can do this with stuff around the house. Dental floss makes a good string.) The point will be to measure the period of the pendulum as accurately and precisely as you can. You will extract a value of the acceleration $g \approx 9.8$ m/s^2 and compare to the accepted value at your latitude, and see if you are within your estimated experimental uncertainties. You will also measure the period as a function of the maximum angle of the swing and compare to the predicted dependence.

Theory of the Simple Plane Pendulum

The pendulum consists of a string of length l and a mass m. We can find the pendulum period in a number of ways, but for our purposes it is best to consider the total energy

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0)$$

where θ_0 is the maximum angle through which the pendulum swings. This gives

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0) = \frac{4g}{l}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

for the angular velocity of the pendulum. If we start the pendulum from rest at $\theta = \theta_0$, the time it takes to swing down to $\theta = 0$ is one quarter of a period T. Noting that $\dot{\theta} < 0$ over this time, we have

$$-\int_{\theta_0}^{0}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)^{-1/2} d\theta = \int_0^{T/4}\sqrt{\frac{4g}{l}}\,dt = \frac{T}{2}\sqrt{\frac{g}{l}}$$

The integral on the left can be transformed to a convenient form using

$$k^2 \equiv \sin^2\frac{\theta_0}{2} \qquad \text{and} \qquad \sin^2 x \equiv \frac{\sin^2\theta/2}{\sin^2\theta_0/2} = \frac{1}{k^2}\sin^2\frac{\theta}{2}$$

The result, which you should prove for yourself, is

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{dx}{(1-k^2\sin^2 x)^{1/2}}$$

This is an "elliptic integral" which cannot be done analytically. However, if $\theta_0 \ll 1$ then $k \ll 1$ and we can expand in small powers of k. The result to first order is

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{4}\sin^2\frac{\theta_0}{2}\right) \tag{1}$$

For small enough θ_0, we neglect the second term and get the result from your high school physics class, namely

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{2}$$

The Measurements

There are two parts to this experiment.

First, you are to extract a value of g by making measurements of the period T and using Equation (2). Measure the period as precisely as you can. For example, use a θ_0 that is small enough so that the correction is negligible, but large enough so that you can easily identify the time of the endpoints of a swing. It is probably a good idea to let the pendulum go through some number N of swings, measuring the total time and dividing by N. Get an idea of the uncertainty by repeating this procedure several times and seeing how different your answers are each time. (You can quote the standard deviation of your set of measurements as one contribution to the uncertainty.)

You can also try making measurements with different lengths of the string, and then use a fit for T as a function of l to determine g, where the "goodness of fit" will give you an idea of the uncertainty in g. (You can do this simply by plotting T^2 versus l in your lab book and finding the slope of the line through the points. You can also do the fit using MATHEMATICA or some other application.)

Look up the expected value of g for your latitude and cite your source in your lab report. How close do you get to the right value? What are the other contributions to the uncertainty? For example, how much is the expected correction from the finite value of θ_0? (Of course, you recorded your value of θ_0 in your lab notebook.) How well can you measure the length l, that is, how well can you determine the endpoints? What about the mass of the string? Is your height above the Earth's surface an important consideration?

The second part of the measurement is to try and confirm Equation (1). For a fixed length, make precise and accurate measurements of the period for different values of θ_0. It would be helpful to calculate the predicted correction first, so you know how precisely you need to make the measurements. Make a plot of the period as a function of θ_0 and compare to (1).