Phys 4702 Intro QM II   Fall 2024   31 Oct 2024

* Happy Halloween!
* Office Hours tomorrow. "Back on Track!"

Review: First Order Time-Dependent Perturbation Theory

$H(t) = H_0 + V(t)$ = "Large" + "Small"

$H_0 |n\rangle = E_n |n\rangle$ is solved

Typical Problem: Initial state $|i\rangle$

↳ what is "transition probability" to state $|n\rangle$?

- Different goal than time-independent P.T.!
- Answers the question re "stationary states"

Found Prob $[i \to n] = |c_n(t)|^2$

$$\text{where } c_n(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_{ni}t'} V_{ni}(t')\,dt'$$ "Transition amplitude"

$\omega_{ni} = (E_n - E_i)/\hbar$ $V_{ni}(t) = \langle n|V(t)|i\rangle$ "Transition matrix element"

Important Application: "Decays"

i.e. System in state $|i\rangle \to$ state $|n\rangle$ in continuum

Better: Final state is eigenstate $|n\rangle$ plus "free" particle(s)

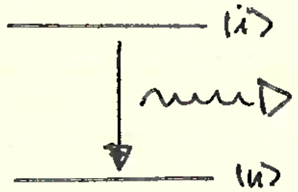
The final eigenstate is $|n\rangle$ w/ $H_0|n\rangle = E_n|n\rangle$

but the full final state includes the continuum of free particle states.

Two Important Examples (Neither of which we will explicitly do.)

1) "Harmonic Perturbation": $V(t) = V_0 e^{i\omega t} + V_0^\dagger e^{-i\omega t}$ Hermitian!

e.g. Electromagnetic decay or absorption of atoms, nuclei



$$c_n(t) = -\frac{i}{\hbar} V_0 \int_0^t \left[e^{i(\omega_{ni}+\omega)t'} + e^{i(\omega_{ni}-\omega)t'} \right] dt'$$

Absorption Emission

$E_n - E_i \approx \hbar\omega$

No Photon! these just become δ-functions!!

Simpler!

2) "Constant Perturbation": $V(t) = V_0$

e.g. Nuclear Beta Decay

$${}^{A}_{N}Z \rightarrow {}^{A}_{N\mp1}(Z\pm1) + e^{\mp} + \overset{(-)}{\nu}_e$$

$|i\rangle \qquad |n\rangle$

$$c_n(t) = -\frac{i}{\hbar} V_{ni} \int_0^t e^{i\omega_{ni}t'} dt'$$

Two Free Particles!

$V_{ni} = \langle n | V_0 (\text{w/ proton} \leftrightarrow \text{neutron}) | i \rangle$

The Truth

- Both examples create / destroy particles
  ↳ Should be doing Quantum Field Theory!
- We will work with "constant" and generic V_{ni}
  ↳ Discover "Fermi's Golden Rule"
- Model for dealing with continuum: One free particle

mass m, momentum $\vec{p} = \hbar\vec{k}$

Non Relativistic

Decays and Fermi's Golden Rule

Simpler for Harmonic !!

Start with $c_n(t) = -\frac{i}{\hbar} V_{ni} \int_0^t e^{i\omega_{ni}t'} dt'$

$$= -\frac{i}{\hbar} V_{ni} \frac{1}{i\omega_{ni}} \left[e^{i\omega_{ni}t} - 1 \right]$$

$$= \frac{V_{ni}}{E_n - E_i} \left[1 - e^{i\omega_{ni}t} \right]$$

Prob $P(i \to N) = \frac{|V_{ni}|^2}{(E_n - E_i)^2} \left[1 - e^{i\omega_{ni}t}\right]\left[1 - e^{-i\omega_{ni}t}\right]$

$$= \frac{4|V_{ni}|^2 t^2}{\hbar^2 (E_n - E_i)^2 t^2 \frac{1}{\hbar^2}} \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right] \quad (*)$$

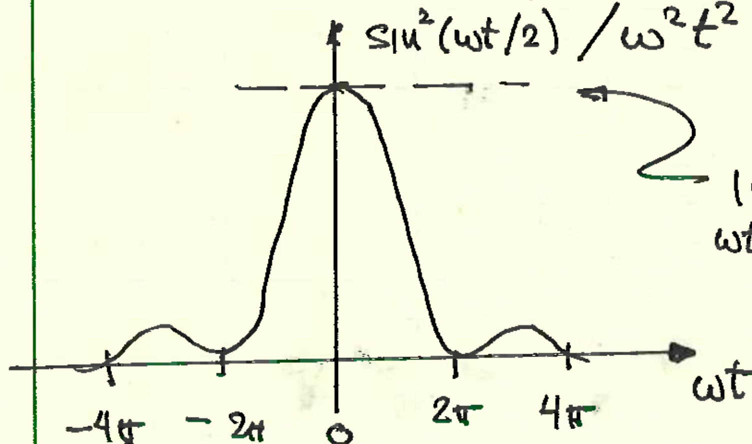
What do we do with this ??

Remember the particle in the continuum!

⇒ Distribution of particles should allow for some energy non-conservation consistent w/ $\Delta E \Delta t \approx \hbar$

Why does that matter? Consider...

(✱) First multiply by t^2/t^2 and write $\omega = (E_n - E_i)/\hbar$



$$\lim_{\omega t \to 0} = \frac{\omega^2 t^2/4}{\omega^2 t^2} = \frac{1}{4}$$

If Energy Conservation were *exact* then $E_n = E_i \Rightarrow \omega t = 0$

$$\Rightarrow P(i \to n) = \frac{1}{\hbar^2} |V_{ni}|^2 t^2$$

i.e. Probability of Decay grows with t^2 !!

i.e. "Decay Rate" dP/dt is not independent of time!

However if Energy Nonconservation is allowed, then we *integrate* over ωt! OK since

$$\Delta(\omega t) \cong 2\pi \text{ means } \frac{\Delta E}{\hbar} \cdot \Delta t \cong 2\pi \text{ or } \Delta E \Delta t \cong \hbar$$

Lots of final states $\Rightarrow$ Average over them!

# of states w/ energy between E & $E+dE$ $= \rho(E)dE$

i.e. $\rho(E) = \frac{dN}{dE}$ $\quad N = \#$ of states @ energy "E"

$$\Rightarrow P(i \to n) = \int_{-\Delta E}^{\Delta E} dE_n \rho(E_n) |c_n(t)|^2$$

$$= 4 \int |V_{ni}|^2 \frac{1}{(E_n - E_i)^2} \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right] \rho(E_n)\, dE_n$$

Take limit as $t \to \infty$. write $x \equiv E_n - E_i$ $\quad \alpha \equiv \frac{t}{2\hbar}$

$$\Rightarrow \lim_{t\to\infty} \frac{1}{(E_n - E_i)^2} \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right] = \lim_{\alpha \to \infty} \frac{\alpha}{\alpha x^2} \sin^2(\alpha x)$$

$$\text{But } \lim_{\alpha \to \infty} \frac{\sin^2(\alpha x)}{\alpha x^2} = \pi \delta(x)$$

PROOF??

$$\Rightarrow P(i\to n) = 4\frac{t}{2\hbar}\,\pi \int |V_{ni}|^2\, \delta(x)\,\rho(x)\,dx$$

$$= \frac{2\pi}{\hbar}\, t\, |V_{ni}|^2\, \rho(E_n = E_i)$$

$\boxed{\text{"Fermi's Golden Rule"}}$

"Energy Conserving δ-Function"

Magic! Averaging over final states gives you $P(i\to n)$ proportional to t not t^2!!

$\Rightarrow$ Concept of "fixed decay rate" Γ

i.e. $\frac{dN}{dt} = -\Gamma N \Rightarrow N(t) = N_0 e^{-\Gamma t}$

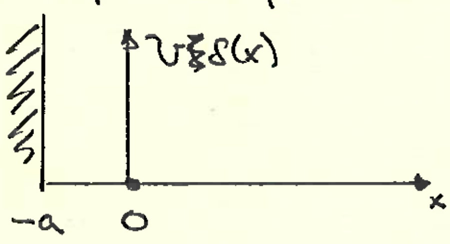
"Exponential Decay": Is it correct!?!?

Interesting Papers / comments

- Double Beta Decay Searches: $\tau = 1/\Gamma > 10^{22}$ years

  But universe is only 10^{10} years old!?

- Simple Example: R. Winter Phys Rev 123 (1961) 1503



$\Rightarrow$ Exponential only for intermediate times!

- Experiment: E.B. Norman, et al. PRL 60 (1988) 2246
  - Test at "long" and "short" times
  - No evidence (yet!) of non-exponential decay.

Example: 1-Body Density of states

Nonrelativistic particle in 3D: H.W. do it for 2D (and fill in blanks for 3D)

"Periodic Boundary conditions"

$u_E(\vec{r}) = N e^{i\vec{k}\cdot\vec{r}}$ Repeats for $x \to x+L$, etc...

3D $k_x = \frac{2\pi}{L} n_x \quad k_y = \frac{2\pi}{L} n_y \quad k_z = \frac{2\pi}{L} n_z$

$$E = \frac{\hbar^2 \vec{k}^2}{2m} = \frac{\hbar^2}{2m} \frac{4\pi^2}{L^2} (n_x^2 + n_y^2 + n_z^2) = \frac{2\pi^2\hbar^2}{mL^2} |\vec{n}|^2$$

$$dE = \frac{4\pi^2\hbar^2}{mL^2} |\vec{n}| \, d|\vec{n}| \qquad dN = 4\pi |\vec{n}|^2 d|\vec{n}|$$

$$\text{3D} \quad \rho(E) = \frac{dN}{dE} = \frac{4\pi |\vec{n}|^2 d|\vec{n}|}{4\pi^2\hbar^2 |\vec{n}| d|\vec{n}| / mL^2} = \frac{mL^2}{\pi\hbar^2} |\vec{n}|$$

$$\text{But } |\vec{n}| = \frac{m^{1/2} L}{\pi\sqrt{2}\,\hbar} E^{1/2}$$

$$\text{3D} \quad \frac{dN}{dE} = \frac{m^{3/2} L^3}{\pi^2 \hbar^3 \sqrt{2}} E^{1/2}$$

this will disappear when you ...

... Normalize wave function.

i.e. $1 = \int_{\text{Box}} u_E^*(\vec{r}) u_E(\vec{r}) \, d^3r = N^2 L^3 \Rightarrow u_E(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}}$