Phys 4702 Intro QM II Fall 2024 29 Oct 2024

* Syllabus Change: this week Time-Dep Pert Theory
* this week HW: Easy! $\Rightarrow$ Catch Up!!

Review: Time Dependent Hamiltonians

$$H(t) = H_0 + V(t) \quad \text{where} \quad H_0|n\rangle = E_n|n\rangle \text{ is solved.}$$

"Interaction Picture"

States: $|\alpha;t\rangle_I \equiv e^{iH_0t/\hbar}|\alpha;t\rangle$ "Schrödinger Picture"

Observables: $A^{(I)}(t) \equiv e^{iH_0t/\hbar} A e^{-iH_0t/\hbar}$

$$\Leftrightarrow \quad i\hbar\frac{d}{dt}|\alpha;t\rangle_I = \underline{\underline{V^{(I)}(t)}}|\alpha;t\rangle_I \quad (*)$$ "Replaces H"

NOTE: If we define $U^{(I)}(t)$ via $|\alpha;t\rangle_I = U^{(I)}(t)|\alpha\rangle$

$$\Leftrightarrow \quad \underline{\underline{i\hbar\frac{d}{dt}U^{(I)} = V^{(I)}(t)U^{(I)}(t)}}$$ will come back to this!

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we used the eigenstates of H_0 as a handy basis:

$$|\alpha;t\rangle_I = \sum_m c_m(t)|m\rangle \Rightarrow c_m(t) = \langle m|\alpha;t\rangle_I$$

Now do $\langle n|$ on $(*)$:

$$\Leftrightarrow \quad i\hbar\dot{c}_n(t) = \langle n|V^{(I)}|\sum_m c_m(t)|m\rangle$$

$$= \sum_m e^{i\omega_{nm}t}V_{nm}c_m(t)$$

$$\omega_{nm} \equiv (E_n - E_m)/\hbar \qquad V_{nm} \equiv \langle n|V(t)|m\rangle$$

Time-Dependent Perturbation Theory

- Everything so far is "exact", i.e. No approximations
- Suppose that $V(t)$ is "small" compared to H_0

$\Rightarrow$ Expect the probability of a "transition" $|i\rangle$ to $|n\rangle$ should be small

i.e. $|c_n(t)|^2 \ll 1$ for $n \neq i$

How to quantify this?

The Cheap Way

$$c_n(t) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \ldots$$

and $V(t) \to \lambda V(t)$

First order: $i\hbar \dot{c}_n^{(1)}(t) = \sum_m e^{i\omega_{nm}t} V_{nm} c_m^{(0)}(0)$

Write $c_n(0) = \delta_{ni}$

$$\Rightarrow i\hbar \dot{c}_n^{(1)}(t) = e^{i\omega_{ni}t} V_{ni}(t)$$

Solve: $$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_{ni}t'} V_{ni}(t')\,dt' + \underline{\text{const}}$$
($=0$ for $n \neq i$)

"Transition Probability for $i \to n$" $\cong |c_n^{(1)}(t)|^2$

Second Order: Friday; Substitute $c_n^{(1)}(t)$ into integral etc...

[We won't pursue this. First order only.]

<u>A More Formal Way</u>: Use $U^{(I)}(t)$ equation instead

$$i\hbar \frac{d}{dt} U^{(I)}(t) = V^{(I)}(t)\, U^{(I)}(t)$$

$$\Leftrightarrow U^{(I)}(t) = \underbrace{U^{(I)}(0)}_{=1} - \frac{i}{\hbar}\int_0^t dt'\, V^{(I)}(t')\, U^{(I)}(t')$$

$$= 1 - \frac{i}{\hbar}\int_0^t dt'\, V^{(I)}(t')\left[1 - \frac{i}{\hbar}\int_0^{t'} V^{(I)}(t'')\, U^{(I)}(t'')\right]$$

$$= \underbrace{1 - \frac{i}{\hbar}\int_0^t dt'\, V^{(I)}(t')}_{\text{First Order}} + \underbrace{\left(-\frac{i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, V^{(I)}(t')\, V^{(I)}(t'')\, U^{(I)}(t'')}_{\text{Higher Orders}} + \dots$$

- Initial state $|i\rangle$ (Eigenstate of H_0 w/ energy E_i)
- Final state $|n\rangle$ " " " " " E_n
- Propagated initial state $|i;t\rangle = U^{(I)}(t)|i\rangle$

$$\cong |i\rangle - \frac{i}{\hbar}\int_0^t dt'\, V^{(I)}(t')\,|i\rangle$$

Probability <u>amplitude</u> to detect system in state $|n\rangle$:

$$\langle n|i;t\rangle = \langle n|i\rangle - \frac{i}{\hbar}\int_0^t dt'\, \langle n|V^{(I)}(t')|i\rangle + \dots$$

$$= \delta_{ni} - \frac{i}{\hbar}\int_0^t dt'\, \langle n|e^{iH_0t'/\hbar}\, V(t')\, e^{-iH_0t'/\hbar}|i\rangle$$

i.e. Prob of "transition" $|i\rangle \to |n\rangle$ is $|\langle n|i;t\rangle|^2$ $\underline{n \neq i}$

$$= -\frac{i}{\hbar}\int_0^t e^{i\omega_{ni}t'} V_{ni}(t')\,dt'$$

$V_{ni} = \langle n|V(t)|i\rangle$

"Transition Matrix Element"

Recall: $\omega_{ni} \equiv (E_n - E_i)/\hbar$

Example: Decaying electric field & charge in SHO

$$H = \underbrace{\frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2}_{H_0 = (N+\frac{1}{2})\hbar\omega} \underbrace{- q x \mathcal{E} e^{-t/\tau}}_{V(t)}$$

$\Rightarrow H_0|n\rangle = (n+\frac{1}{2})\hbar\omega\,|n\rangle$

$n = 0, 1, 2, \ldots$

Problem: Oscillator in ground state $|0\rangle$ @ $t=0$

$V(t)$ "switched on" @ $t=0$, then decays away.

$\Rightarrow$ What is the probability that you find it in $|1\rangle$ at $t \to \infty$?

Just use the formula!

$$\langle 1|0;t\rangle = -\frac{i}{\hbar}\int_0^t dt'\, e^{i\omega_{10}t'} V_{10}(t')$$

$\omega_{10} = (E_1 - E_0)/\hbar = [(1+\frac{1}{2}) - \frac{1}{2}]\omega = \omega$

$V_{10} = \langle 1|(-q x \mathcal{E} e^{-t/\tau})|0\rangle$

$= -q\mathcal{E}\langle 1|x|0\rangle e^{-t/\tau}$

Have $x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger) \Rightarrow \langle 1|x|0\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$

$$\Rightarrow \langle 1|0;t\rangle = i\,\frac{q\mathcal{E}}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\int_0^{\cancel{t}\to\infty} e^{i\omega t' - t'/\tau}\,dt'$$

$$= i\,\frac{q\mathcal{E}}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(-\frac{1}{i\omega - 1/\tau}\right) \times \frac{\tau}{\tau}$$

So Probability $= \frac{q^2\mathcal{E}^2}{2m\hbar\omega}\;\frac{\tau^2}{1+\omega^2\tau^2}$ Sanity check: $\tau \to 0 \Rightarrow$ Prob $= 0$

Interesting question: Under what condition is Prob $\ll 1$??

Constant Perturbation

$V(t) = 0$ for $t < 0$ $= V_0$ for $t \geq 0$

Model for radioactive decay, etc...

$\Rightarrow$ "Fermi's Golden Rule" (invented by Dirac)

Introduce now, finish on Thursday.

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_{ni}t'}\,V_{ni}\,dt' \qquad V_{ni} = \text{constant}$$

$$= -\frac{i}{\hbar}V_{ni}\int_0^t e^{i\omega_{ni}t'}\,dt'$$

Multiply by $\hbar$

$$= \frac{V_{in}}{(E_n - E_i)}\left[1 - e^{i\omega_{ni}t}\right]$$

Reversed order of integration limits

$$\text{Prob}(i \to n) = |c_n^{(1)}(t)|^2 = \frac{|V_{in}|^2}{(E_n - E_i)^2}\,|(1 - e^{i\omega_{ni}t})|^2$$

But $|1-e^{i\omega_{ni}t}|^2 = (1-e^{i\omega_{ni}t})(1-e^{-i\omega_{ni}t})$

$$= 1 - 2\cos(\omega_{ni}t) + 1 = 2\,[1-\cos\omega_{ni}t]$$

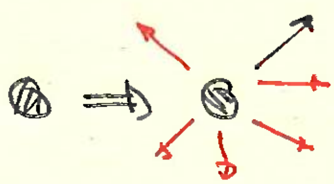
$$= 4\sin^2\frac{\omega_{ni}t}{2}$$

$$\Rightarrow \text{Prob}\,(i\to n) = \frac{4\,|V_{in}|^2}{(E_n-E_i)^2}\sin^2\left[\frac{(E_n-E_i)t}{2\hbar}\right]$$

What does this mean?

How can we use it?

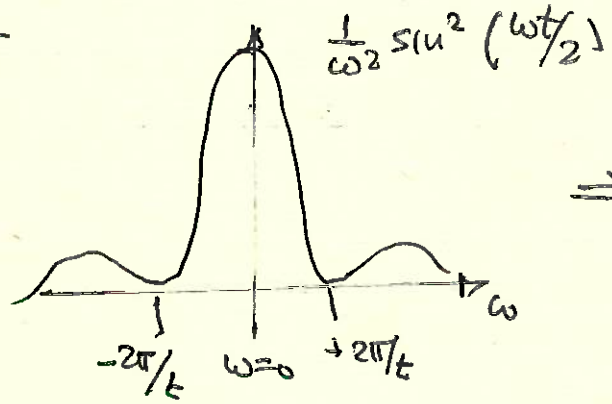
"Decay":



Lots of different final states!!

We need to account for the continuum of "$|n\rangle$"s!

Hint



$\Rightarrow$ Include all states within ΔE where $\Delta E\,\Delta t \sim \hbar$

$\Rightarrow$ Will get transition probability in terms of "Density of States"