Phys 4702 Intro QM II Fall 2024 24 Oct 2024

* Next HW will be "simple basics": Catch Up!
* Office hours tomorrow!

Review: Interaction Picture

"Time Dependent Hamiltonian" $H = H_0 + V(t)$

States: $|\alpha;t\rangle_I \equiv e^{iH_0t/\hbar}|\alpha;t\rangle$

Observables: $A^{(I)} \equiv e^{iH_0t/\hbar} A e^{-iH_0t/\hbar}$

$$\Leftrightarrow \quad i\hbar\frac{d}{dt}|\alpha;t\rangle_I = V^{(I)}(t)|\alpha;t\rangle_I$$

Recall: $i\hbar\frac{d}{dt}|\alpha;t\rangle = H(t)|\alpha;t\rangle$ ("similar")

"Expand in H_0 Eigenstates": $H_0|n\rangle = E_n|n\rangle$

Project: $i\hbar\frac{d}{dt}\langle n|\alpha;t\rangle_I = \langle n|V^{(I)}(t)|\alpha;t\rangle_I$

"Differential Equation"

Expand: $1 = \sum_m |m\rangle\langle m|$

$$\Leftrightarrow \quad |\alpha;t\rangle_I = 1|\alpha;t\rangle_I = \sum_m |m\rangle\langle m|\alpha;t\rangle_I$$

Write: $c_n(t) \equiv \langle n|\alpha;t\rangle_I = e^{iE_nt/\hbar}\langle n|\alpha;t\rangle$

i.e. $|c_n(t)|^2 = |\langle n|\alpha;t\rangle|^2$

= Probability of finding system in eigenstate $|n\rangle$ @ time t

[Didn't point this out in class on Tuesday.]

Now we can make things look simple:

$$i\hbar \frac{d}{dt}\Big[\sum_m c_m(t) \underbrace{\langle n|m\rangle}_{\delta_{nm}}\Big] = \sum_m \underbrace{\langle n|V^{(I)}|m\rangle}_{(*)} \underbrace{\langle m|\alpha_i t\rangle_I}_{(**)}$$

$$(*) = \langle n| e^{iH_0 t/\hbar} V(t) e^{-iH_0 t/\hbar} |m\rangle$$
$$= e^{i(E_n - E_m)t/\hbar} \langle n|V(t)|m\rangle \equiv e^{i\omega_{nm} t} V_{nm}$$

$(**) = c_m(t) \Rightarrow$ "Coupled First Order ODE's"

$$i\hbar \dot{c}_n(t) = \sum_m V_{nm} e^{i\omega_{nm} t} c_m(t)$$

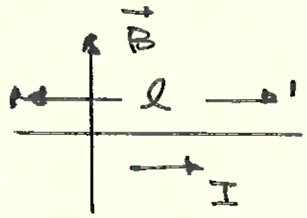
"Jumping off point for Time-Dep. Pert Theory"

But we can use this formalism directly for a large class of problems!

Magnetic Resonance

- "Holding Field" (in z-direction) with "Oscillating Field" (in x-direction) $\Rightarrow$ Spin-Flipping
- Very sensitive to frequency of oscillations
  $\Rightarrow$ Useful technique! (Several Nobel prizes!)
- Most Familiar to you: "MRI"
  Sensitive to proton spin (i.e. living tissue!) in different magnetic environments.
- First we'll do a classical discussion, then all HW: Derive "Rabi's Formula"

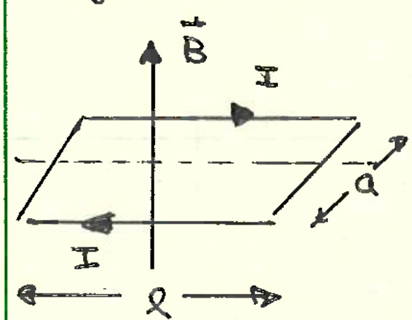
Magnetic Force on a Wire   (Same Physics II stuff)



$$d\vec F = \frac{1}{c}dq\,\vec v\times\vec B = \frac{1}{c}dq\,\frac{d\vec\ell}{dt}\times\vec B = \frac{1}{c}I\,d\vec\ell\times\vec B$$

$$\Rightarrow\ \vec F = \frac{I}{c}\vec\ell\times\vec B$$

Magnetic Torque on a Current Loop



$$\vec F = \frac{I}{c}\vec\ell\times\vec B = 0 \text{ on sides}$$

$$= \text{Equal \& Opposite on "top" \& "bottom"}$$

$$\Rightarrow\ \vec\tau = 2\,\frac{\vec a}{2}\times\vec F = \frac{I}{c}\vec a\times(\vec\ell\times\vec B)$$

Identity: $\vec A\times(\vec B\times\vec C) + \vec B\times(\vec C\times\vec A) + \vec C\times(\vec A\times\vec B) = 0$

$$\Rightarrow\ \vec a\times(\vec\ell\times\vec B) = -\vec\ell\times(\underbrace{\vec B\times\vec a}_{\text{Parallel to }\vec\ell}) - \vec B\times(\vec a\times\vec\ell)$$

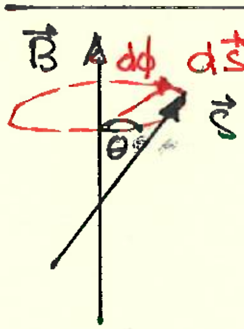
$$\Rightarrow\ \vec\tau = \frac{I}{c}(\vec a\times\vec\ell)\overset{=0}{\times}\vec B \quad \text{(reverse order changes sign)}$$

But $|\vec a\times\vec\ell| = a\ell$ = area of loop

and $\vec a\times\vec\ell$ perpendicular to plane of loop

i.e. $\vec\mu = \frac{I}{c}(\vec a\times\vec\ell) \Rightarrow \boxed{\vec\tau = \vec\mu\times\vec B}$

Precession of Magnetic Dipole   "Larmor Frequency"



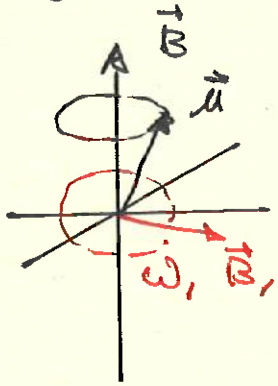
$d\vec s = \vec\tau\,dt$

i.e. $|d\vec s| = |\vec s|\sin\theta\,d\phi$

write $\vec\mu = \gamma\vec s$ $= |\vec\mu\times\vec B|\,dt = \gamma|\vec s|B\sin\theta\,dt$

$$\Rightarrow\ \omega = \frac{d\phi}{dt} = \gamma B$$

Magnetic Resonance: Classical

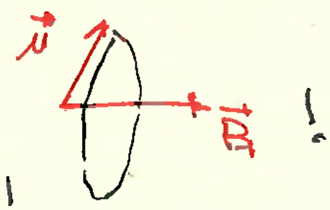


- Precession at Larmor frequency ω
- $\vec{B}_1$ field in xy plane rotating with angular frequency ω_1

Take $B_1 \ll B$

↳ Expect almost no effect unless...

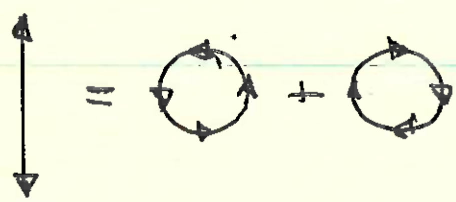
... $\omega_1 = \omega$!!



↳ In B_1 rotating frame you see

i.e. Rotate about B_1! "spin flip" !!

But how to make a rotating field $\vec{B}_1$? Easy!!



Linear oscillating $\vec{B}$ field creates two rotating fields w/ $\omega = \pm\omega_1$!!

Quantum Mechanical Treatment

$$H = -\vec{\mu}\cdot\vec{B} \qquad \vec{B} = B_0\hat{z} + B_1\cos\omega_1 t\ \hat{x} \qquad B_1 \ll B_0$$

"two rotating fields"

$$= B_1(\cos\omega_1 t\ \hat{x} + \sin\omega_1 t\ \hat{y})/2 \quad \text{"}\omega_1\text{"}$$
$$+ B_1(\cos\omega_1 t\ \hat{x} - \sin\omega_1 t\ \hat{y})/2 \quad \text{"}-\omega_1\text{"}$$

Have $\vec{\mu} = \gamma\vec{S}$ ⟹ general problem!

For "spin-1/2" we write

$$\vec{\mu} = g \times \text{"magneton"} \times \vec{S} \quad \text{w/ "magneton"} = \frac{e\hbar}{2Mc}$$

BTW $g(\text{electron}) = 2 + \frac{\alpha}{2\pi} + \cdots$ "Fundamental"

$g(\text{proton}) = 2.79$ "Not!"

$g(\text{neutron}) = -1.9$ "composite!!"

Formulation for spin-1/2

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot[B_0\hat{z} + B_1\cos\omega_1 t\,\hat{x}]$$
$$= -\gamma B_0 S_z - \gamma B_1(\cos\omega_1 t) S_x$$

[NOTE FOR HW: $-\gamma B_0 \to \omega_0$ $\quad -\gamma B_1 \to \omega_1$ $\quad \omega_1 \to \omega$]

The Problem: Find prob for "spin-down" as a function of time if "spin-up" @ $t=0$

$\Rightarrow$ Watch what happens if you vary ω_1!!

Expect something dramatic when $\omega_1 \approx \omega_{LARMOR}$

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i.e. (Back to formalism!) Find $|c_-(t)|^2 = |c_2(t)|^2$

when $c_+(t) = c_1(t) = 1$ at $t=0$.

with $i\hbar\dot{c}_n(t) = \sum_m V_{nm}\, e^{i\omega_{nm}t} c_m(t)$ For $n,m = 1 \,\&\, 2$

Homework Notation: $H = \omega_0 S_z + \omega_1 \cos\omega t\, S_x$

$= H_0 + V(t)$

States are $|\pm\hat{z}\rangle$ w/ $H_0|\pm\hat{z}\rangle = (\pm\hbar\omega_0/2)|\pm\hat{z}\rangle$

$\Rightarrow E_1 = \frac{1}{2}\hbar\omega_0 \quad E_2 = -\frac{1}{2}\hbar\omega_0$

i.e. $\omega_{12} = \omega_0 \quad \omega_{21} = -\omega_0$

$= \frac{E_1 - E_2}{\hbar}$

<u>Also</u> $V(t) = \omega_1 \cos(\omega t)\, S_x$

<u>Recall</u>: $S_z \doteq \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad S_x \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now write down the <u>two</u> coupled ODE's:

$$i\hbar \dot{c}_1 = V_{11} e^{i(0)t} c_1 + V_{12} e^{i(\omega_{12})t} c_2$$
$$i\hbar \dot{c}_2 = V_{21} e^{i(\omega_{21})t} c_1 + V_{22} e^{i(0)t} c_2$$

But $V_{11} = 0 = V_{22}$ $\quad V_{12} = \frac{\hbar}{2}\omega_1 \cos(\omega t) = V_{21}$

$\Rightarrow$ $i\hbar \dot{c}_1 = \frac{\hbar}{2}\omega_1 e^{i\omega_0 t} \cos(\omega t)\, c_2$

$i\hbar \dot{c}_2 = \frac{\hbar}{2}\omega_1 e^{-i\omega_0 t} \cos(\omega t)\, c_1$

with $c_1(0) = 1 \quad c_2(0) = 0$

Now go to town on it!
Remember that you want to find $|c_2(t)|^2$

$\Rightarrow$ <u>Rabi's Formula</u> Typo in problem? Must check!!

and plot $|c_2(t)|^2$ versus ω for $\frac{B_x}{B_z} = \frac{\omega_1}{\omega_0} = \frac{1}{1000}$

$\Rightarrow$ "Sharp Resonance"