Phys 4702 Intro QM II Fall 2024 21 Nov 2024

* Still working on "Final" HW: Hang on, Pay Attention!
* Office Hours tomorrow? Anyone around?

Review: Free Particle Dirac Hamiltonian

$$\underline{\underline{H}}^* = \vec{\underline{\underline{\alpha}}}^* \cdot \vec{p} + \underline{\underline{\beta}}^* m \, \mathbb{1}$$

* Matrices in 4D "spinor space" [Don't confuse w/ 4D spacetime!]

Operator (Identity op.)

Energy Eigenstates: $\underline{\underline{H}} |\underline{\psi}\rangle = E |\underline{\psi}\rangle$ 4D Column vector

then do $\langle \vec{r}\,'|$ on both sides. (i.e. Position eig. $|\vec{r}\,'\rangle$)

$$\Leftrightarrow \; i \frac{\partial}{\partial t} \underline{\psi}(\vec{r}\,', t) = -i \vec{\underline{\underline{\alpha}}} \cdot \vec{\nabla} \underline{\psi}(\vec{r}\,', t) + \underline{\underline{\beta}} m \psi(\vec{r}\,', t).$$

Put $\psi(\vec{r}\,', t) = e^{-iEt} \psi(\vec{r}\,') = \underline{N} e^{-iEt} e^{i \vec{r}\,' \cdot \vec{p}\,'}$ ← Eigenvalue

Plane wave

$$\Leftrightarrow \; E \underline{N} = \vec{\underline{\underline{\alpha}}} \cdot \underline{N} \, \vec{p}\,' + \underline{\underline{\beta}} m \underline{N}$$

You can write this as an eigenvalue equation in spinor space:

$$(\vec{\underline{\underline{\alpha}}} \cdot \vec{p}\,' + \underline{\underline{\beta}} m) \underline{N} = E \underline{N}$$

$\Rightarrow$ Find (4) eigenvalues E and (4) eigenvectors $\underline{N}$

Recall: Major Triumph of Dirac Equation

$\rho \equiv \psi^\dagger \psi$ is positive definite and

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \quad \text{w/} \quad \vec{j} \equiv \psi^\dagger \vec{\alpha} \psi \quad \text{Conserved!}$$

Single Particle Wave Functions

NOTE: Details will be a Homework Problem!

Need to pick a suitable $\vec{\underline{\underline{\alpha}}}$, $\underline{\underline{\beta}}$ matrix formulation.
[the γ^μ anti commutation relations must be satisfied.]

We choose to make $\vec{\alpha}, \beta$ Hermitian. We will use

$$\vec{\underline{\underline{\alpha}}} = \begin{bmatrix} \underline{\underline{0}} & \vec{\underline{\underline{\sigma}}} \\ \vec{\underline{\underline{\sigma}}} & \underline{\underline{0}} \end{bmatrix} \qquad \underline{\underline{\beta}} = \begin{bmatrix} \underline{\underline{1}} & \underline{\underline{0}} \\ \underline{\underline{0}} & -\underline{\underline{1}} \end{bmatrix}$$

HW: Show these have the right properties.

Let particle move in z-direction i.e. $\vec{p}' = p\hat{z}$

$$\Leftrightarrow \; (p\underline{\underline{\alpha}}_z + m\underline{\underline{\beta}})\underline{N} = E\underline{N}$$

i.e.
$$\begin{bmatrix} m & 0 & p & 0 \\ 0 & m & 0 & -p \\ p & 0 & -m & 0 \\ 0 & -p & 0 & -m \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = E \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \qquad \text{for } \underline{N} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

Find: $E_{1,2} = (p^2+m^2)^{1/2} \equiv +E_p$

$E_{3,4} = -E_p$ "Eigenvalues"

"Normalized Eigenvectors"

$$\underline{u}_R^{(+)}(p) = \begin{bmatrix} 1 \\ 0 \\ p/(E_p+m) \\ 0 \end{bmatrix} \qquad \underline{u}_L^{(+)}(p) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -p/(E_p+m) \end{bmatrix} \qquad E = +E_p$$

$$\underline{u}_R^{(-)}(p) = \begin{bmatrix} -p/(E_p+m) \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{u}_L^{(-)}(p) = \begin{bmatrix} 0 \\ p/(E_p+m) \\ 0 \\ 1 \end{bmatrix} \qquad E = -E_p$$

$$\vec{\underline{\underline{\Sigma}}}\cdot\hat{p}\,\underline{u}_R = +\underline{u}_R \qquad \vec{\Sigma} = \begin{bmatrix} \vec{\underline{\underline{\sigma}}} & 0 \\ 0 & \vec{\underline{\underline{\sigma}}} \end{bmatrix} \qquad \vec{\underline{\underline{\Sigma}}}\cdot\hat{p}\,\underline{u}_L = -\underline{u}_L$$

Non-Relativistic Reduction

Do we get the (non-relativistic) Schrödinger Equation when we have kinetic energy $K \ll m$??

First write $\underline{\underline{H}}\,\underline{\psi} = E\underline{\psi}$ as $\begin{bmatrix} \underline{\underline{m}} & \underline{\underline{\vec{\sigma}}}\cdot\vec{p} \\ \underline{\underline{\vec{\sigma}}}\cdot\vec{p} & -\underline{\underline{m}} \end{bmatrix}\begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} = E\begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix}$

i.e. Expressed using 2D spinors [Could write $|\psi\rangle$ but I won't.]

Now $E = K + m \approx m$ for $K \ll m$.

$\Rightarrow$ "Bottom" equation gives $(\underline{\underline{\vec{\sigma}}}\cdot\vec{p})\,\underline{u} - m\underline{v} = m\underline{v}$

i.e. $(\underline{\underline{\vec{\sigma}}}\cdot\vec{p})\,\underline{u} = 2m\underline{v}$

"upper" equation is $m\underline{u} + (\underline{\underline{\vec{\sigma}}}\cdot\vec{p})\,\underline{v} = E\underline{u} = (K+m)\underline{u}$

i.e. $(\underline{\underline{\vec{\sigma}}}\cdot\vec{p})\,\underline{v} = \frac{1}{2m}(\underline{\underline{\vec{\sigma}}}\cdot\vec{p})(\underline{\underline{\vec{\sigma}}}\cdot\vec{p})\,\underline{u} = K\underline{u}$ $(*)$

NOTE: Did we prove this last semester?

$$
\begin{aligned}
(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) &= \sigma_i a_i \sigma_j b_j \\
&= \left(\tfrac{1}{2}\{\sigma_i,\sigma_j\} + \tfrac{1}{2}[\sigma_i,\sigma_j]\right) a_i b_j \\
&= (\delta_{ij} + i\varepsilon_{ijk}\sigma_k)\, a_i b_j \\
&= a_i b_i + i\sigma_k \varepsilon_{ijk} a_i b_j \\
&= \vec{a}\cdot\vec{b} + i\sigma_k (\vec{a}\times\vec{b})_k \\
&= \vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})
\end{aligned}
$$

$\Rightarrow$ $(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) = \vec{p}^{\,2} + i\,\vec{\sigma}\cdot\underline{\underline{(\vec{p}\times\vec{p})}}$

$= 0$ (p_i <u>commute</u> !!)

i.e. $\dfrac{\vec{p}^{\,2}}{2m}\,\underline{u} = k\underline{u}$ Correct!!

Same equation for $\underline{u} = \begin{bmatrix}1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\end{bmatrix}$ as well!!

<u>Adding Magnetism</u>

<u>Recall</u> (10 Sep) : $\vec{p} \longrightarrow \vec{p} - q\vec{A} \equiv \tilde{p}$ (For convenience)

$\Rightarrow$ (*) $\dfrac{1}{2m}(\vec{\sigma}\cdot\tilde{p})(\vec{\sigma}\cdot\tilde{p})\,\underline{u} = k\underline{u}$

Now be careful ! $\tilde{p}\times\tilde{p} \neq 0$

i.e. $\dfrac{\tilde{p}^{\,2}}{2m}\underline{u} + \underline{\underline{i\dfrac{\vec{\sigma}}{2m}\cdot(\tilde{p}\times\tilde{p})}}\,\underline{u} = k\underline{u}$ (**)

what is this ?

Have $\tilde{p} = \vec{p} - q\vec{A}(\vec{r}) \Rightarrow$ Could use $[\vec{p}, F(\vec{r})]$ etc...

but let's just use coord rep

i.e. $\tilde{p}\times\tilde{p}u = (\vec{p} - q\vec{A})\times(\vec{p} - q\vec{A})u$

$\rightarrow (-i\vec{\nabla} - q\vec{A})\times(-i\vec{\nabla} - q\vec{A})u$ Cross terms left so...

$= iq\,\vec{\nabla}\times(\vec{A}u) + iq\,\vec{A}\times\vec{\nabla}u$

$= iq\,\underbrace{(\vec{\nabla}\times\vec{A})}_{=\vec{B}}u + \underbrace{iq\,\vec{\nabla}u\times\vec{A} + iq\,\vec{A}\times\vec{\nabla}u}_{=0}$

Magnetic Field!!

$$\Leftrightarrow (**) \quad \frac{\vec{P}^2}{2m}\underline{u} + \underbrace{\frac{1}{2m} i\vec{\sigma}\cdot(iq\vec{B})}\underline{u} = K\underline{u}$$

$$= -\frac{q}{2m}\vec{\sigma}\cdot\vec{B} = -\frac{2q}{2m}\frac{\vec{\sigma}}{2}\cdot\vec{B}$$

i.e. there is an "extra" term equal to

$$-\vec{\mu}\cdot\vec{B} \quad \text{w/} \quad \mu \equiv g\frac{q}{2m}\vec{S} \quad \text{w/} \quad \vec{S} = \frac{1}{2}\vec{\sigma} \quad \& \quad g = 2$$

Wow! Absorb this! This was our course on Day #1!!

Angular Momentum

Is the Dirac Hamiltonian rotationally invariant?

It should be, but...

$$\begin{aligned}[\vec{\alpha}\cdot\vec{p}, L_i] &= [\alpha_\ell p_\ell, \varepsilon_{ijk} r_j p_k] \\ &= \varepsilon_{ijk}\alpha_\ell [p_\ell, r_j] p_k \\ &= \varepsilon_{ijk}\alpha_\ell(-i\delta_{\ell j}) p_k \\ &= -i\varepsilon_{ijk}\alpha_j p_k \neq 0\end{aligned}$$

What's going on? Doesn't $\vec{L}$ generate rotations??

No, It doesn't! But consider

$$\vec{\Sigma} = \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \vec{\Sigma}\beta = \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{bmatrix} = \beta\vec{\Sigma}$$

and

$$[\alpha_i, \Sigma_j] = \left[\begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \begin{bmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{bmatrix}\right] = \begin{bmatrix} 0 & \sigma_i\sigma_j \\ \sigma_i\sigma_j & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma_j\sigma_i \\ \sigma_j\sigma_i & 0 \end{bmatrix}$$

i.e. $[\alpha_i, \Sigma_j] = \begin{bmatrix} 0 & 2i\varepsilon_{ijk}\sigma_k \\ 2i\varepsilon_{ijk}\sigma_k & 0 \end{bmatrix} = 2i\varepsilon_{ijk}\alpha_k$

$\Leftrightarrow [\vec{\alpha}\cdot\vec{p}, \Sigma_j] = [\alpha_i, \Sigma_j]p_i = 2i\varepsilon_{ijk}\alpha_k p_i$
$= 2i\varepsilon_{jki}\alpha_k p_i$

aka $[\vec{\alpha}\cdot\vec{p}, \Sigma_i] = 2i\varepsilon_{ijk}\alpha_j p_k$

$\Leftrightarrow [\vec{\alpha}\cdot\vec{p}, L_i + \frac{1}{2}\Sigma_i] = 0$

i.e. Rotations generated by $\vec{J} \equiv \vec{L} + \frac{1}{2}\vec{\Sigma}$
$= \vec{L} + \vec{S}$

"Total Angular Momentum"

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* Plan for last two classes:
Fermions & Bosons $\Rightarrow$ Klein-Gordon Field

* Will post final HW ASAP