Phys 4702 Intro QM II Fall 2024 19 Nov 2024

* Final HW assignment: Will make it up soon
* Will be revising syllabus for final week

Review: Klein-Gordon Equation

$$[\partial_\mu \partial^\mu + m^2]\, \psi(x^\mu) = \left[\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right] \psi(\vec{x}, t) = 0$$

Free particle with mass m Covariant!

and $\psi(x^\mu) = N e^{-i p_\mu x^\mu} \Rightarrow p_\mu p^\mu + m^2 = -E^2 + \vec{p}^{\,2} + m^2 = 0$

i.e. $E^2 = \vec{p}^{\,2} + m^2$

- Correct energy eigenvalues!
- But must admit negative energies $E = -(\vec{p}^{\,2} + m^2)^{1/2}$
- And can't define positive-definite ρ !!
  (However, hints of "antiparticles" ...)

Today: Dirac Equation

Don't like second order in time $\Rightarrow$ Find wave eq in 1st order!

But it needs to be covariant $\Rightarrow$ First order in space, too!

$$\Rightarrow (i \gamma^\mu \partial_\mu - m)\, \psi(x^\alpha) = 0$$

"Just some values"

could be any constant but it will turn out to be mass!

Must give correct energy eigenvalues!

$\Rightarrow$ Force it to be consistent with K.G. Equation!!

"Easy" to do: $-(+i\gamma^\nu\partial_\nu+m)(i\gamma^\mu\partial_\mu-m)\psi=0$

$$-(-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu-m^2)\psi=0$$

i.e. $(\underline{\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu}+m^2)\psi=0$

switch dummies — this must equal $\partial_\mu\partial^\mu=\eta^{\mu\nu}\partial_\mu\partial_\nu$

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write $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu^{(*)}=\eta^{\mu\nu}\partial_\mu\partial_\nu$

$$=\gamma^\mu\gamma^\nu\underline{\partial_\nu\partial_\mu}=\underline{\gamma^\nu\gamma^\mu\partial_\mu\partial_\nu}^{(*)}$$

switch order — switch dummies

Now add: $[\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu]\partial_\mu\partial_\nu=2\eta^{\mu\nu}\partial_\mu\partial_\nu$

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i.e. Require that $\frac{1}{2}[\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu]\equiv\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}=\eta^{\mu\nu}$

where $\{A,B\}\equiv AB+BA$ "anti-commutator"

Recall $\eta^{\mu\nu}=\begin{bmatrix}1&&&0\\&-1&&\\&&-1&\\0&&&-1\end{bmatrix}\Rightarrow(\gamma^0)^2=1\quad(\gamma^i)^2=-1\quad i=1,2,3$

and $\gamma^\mu\gamma^\nu=-\gamma^\nu\gamma^\mu\ ^{(**)}\quad\mu\neq\nu$

"Clifford Algebra" (1873) aka "Bi-Quaternions"

- Obviously the γ^μ are not just complex numbers!
- Pauli matrices? Maybe $\gamma^0=\mathbb{1}$ and $\gamma^j=i\sigma_j$? Complete!!

  No! $\mathbb{1}(i\sigma_j)\neq-(i\sigma_j)\mathbb{1}$ "Not big enough"
- 3×3 matrices? Apparently not (but I don't know why)
- You can make it work with 4×4! (We'll see soon.)

But these are choices! (Not unique!) However $(**)\Rightarrow\mathrm{Tr}\,\gamma^\mu=0$

Before finding an explicit form of γ^μ, find Hamiltonian.

$$(i\gamma^\mu \partial_\mu - m)\, e^{-ix^\mu p_\mu} = 0 \;\Rightarrow\; i\gamma^\mu(-ip_\mu) - m = 0$$

i.e. $\gamma^\mu p_\mu - m = \gamma^0 E - \vec{\gamma}\cdot\vec{p} - m = 0$

$\times\gamma^0 \;\Rightarrow\; E = \gamma^0\vec{\gamma}\cdot\vec{p} + \gamma^0 m$ ($\vec{p}$ is just a vector)

$\Leftrightarrow\; H = \vec{\alpha}\cdot\vec{p} + \beta m$ (Now $\vec{p}$ is an operator)

$\vec{\alpha} \equiv \gamma^0\vec{\gamma} \qquad \beta \equiv \gamma^0$ (Dirac's starting point)

Free Particle! (will add electromagnetism later via A^μ)

One (non-universal) convention: Make $\vec{\alpha}$ and β Hermitian.

$\Leftrightarrow\; \gamma^{0\dagger} = \gamma^0$ But $\vec{\alpha}^\dagger = (\gamma^0\vec{\gamma})^\dagger = \vec{\gamma}^\dagger\gamma^0$

$= \vec{\alpha} = \gamma^0\vec{\gamma} = -\vec{\gamma}\gamma^0$ i.e. $\vec{\gamma}^\dagger = -\vec{\gamma}$

"γ^0 is Hermitian but $\vec{\gamma}$ is anti-Hermitian"

There are still choices for the γ^μ but we'll use these:

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \gamma^0 \qquad \text{"}2\times2\text{'s of }2\times2\text{'s"}$$

$\Rightarrow \vec{\gamma} = \gamma^0\vec{\alpha}$.

Example: $\gamma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{bmatrix} \qquad \gamma_y = \begin{bmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{bmatrix}$

so $\gamma_x\gamma_y = \begin{bmatrix} -\sigma_x\sigma_y & 0 \\ 0 & -\sigma_x\sigma_y \end{bmatrix} \qquad \gamma_y\gamma_x = \begin{bmatrix} -\sigma_y\sigma_x & 0 \\ 0 & -\sigma_y\sigma_x \end{bmatrix} = -\gamma_x\gamma_y$

since $\sigma_x\sigma_y = -\sigma_y\sigma_x$

The Conserved Current

First consider the Schrödinger Equation:

$$\cancel{i\partial_t \psi} \quad H = \vec{\alpha}\cdot\vec{p} + \beta m$$

$$\Rightarrow i\partial_t \psi = -i\vec{\alpha}\cdot\vec{\nabla}\psi + \beta m \psi \quad (*)$$

But $\vec{\alpha}$ and β are 4×4 matrices

$\Rightarrow \psi$ is a four-component column vector! (more later)

We used to write $\rho = \psi^*\psi$ for the probability density.

$\Rightarrow$ Does it make sense to now write $\rho = \psi^\dagger\psi$??

Try it! $\frac{\partial\rho}{\partial t} = (\partial_t \psi^\dagger)\psi + \psi^\dagger(\partial_t \psi)$ From $(*)$

$$\partial_t \psi = -\vec{\alpha}\cdot\vec{\nabla}\psi - i\beta m\psi$$

Recall: $\vec{\alpha}, \beta$ are Hermitian

also $\partial_t \psi^\dagger = -\vec{\nabla}\psi^\dagger\cdot\vec{\alpha} + i\beta m\psi^\dagger$

$$\Rightarrow \frac{\partial\rho}{\partial t} = -\vec{\nabla}\psi^\dagger\cdot\vec{\alpha}\psi - \psi^\dagger\vec{\alpha}\cdot\vec{\nabla}\psi = -\vec{\nabla}\cdot(\psi^\dagger\vec{\alpha}\psi)$$

$$= -\vec{\nabla}\cdot\vec{j} \quad \text{where} \quad \vec{j} \equiv \psi^\dagger\vec{\alpha}\psi$$

we have a conserved current!

NOTE: Any Hermitian "potential" would also cancel.

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i.e. $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$ But can we make this covariant?

We can make it look covariant with $\bar{\psi} \equiv \psi^\dagger\beta = \psi^\dagger\gamma^0$

$$\Rightarrow \rho = \psi^\dagger\psi = \cancel{(\bar{\psi}\gamma^0)} = \psi^\dagger\gamma^0\gamma^0\psi = \bar{\psi}\gamma^0\psi$$

and $\vec{j} = \psi^\dagger\vec{\alpha}\psi = \psi^\dagger\gamma^0\gamma^0\vec{\alpha}\psi = \bar{\psi}\underline{\underline{\gamma^0\vec{\alpha}}}\psi = \bar{\psi}\vec{\gamma}\psi$

[It's subtle to prove covariance] (Recall: $= \vec{\gamma}$) $\Rightarrow j^\mu = \bar{\psi}\gamma^\mu\psi$

Free Particle Solutions

Recall energy eigenvalues $E = \gamma^0 \vec{\gamma} \cdot \vec{p} + \gamma^0 m$

Particle at rest $(\vec{p} = 0)$

$$\Leftrightarrow i\partial_t \psi = E\psi \Rightarrow i\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} m & & & \\ & m & & \\ & & -m & \\ & & & -m \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Four independ equations $\Rightarrow$ Four independent solutions!

$$\text{i.e. } \psi = e^{-imt}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad = e^{-imt}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad = e^{+imt}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad = e^{+imt}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Positive Energy — Negative Energy

What do you think the two components turn out to be?

Particle with constant momentum $(\vec{p} = p\hat{z})$

i.e. want to solve $H\psi = E\psi$ for $H = \alpha_z p + \beta m$

$$\Leftrightarrow \text{Find eigenvalues for } \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix} p + \begin{bmatrix} m & 0 \\ 0 & -m \end{bmatrix}$$

$$= \begin{bmatrix} m & 0 & p & 0 \\ 0 & m & 0 & -p \\ p & 0 & -m & 0 \\ 0 & -p & 0 & -m \end{bmatrix}$$

coupled

For "1" and "3" find $E = \pm E_p$

"2" and "4" find same

i.e. upper two are positive energy

(lower) (negative)

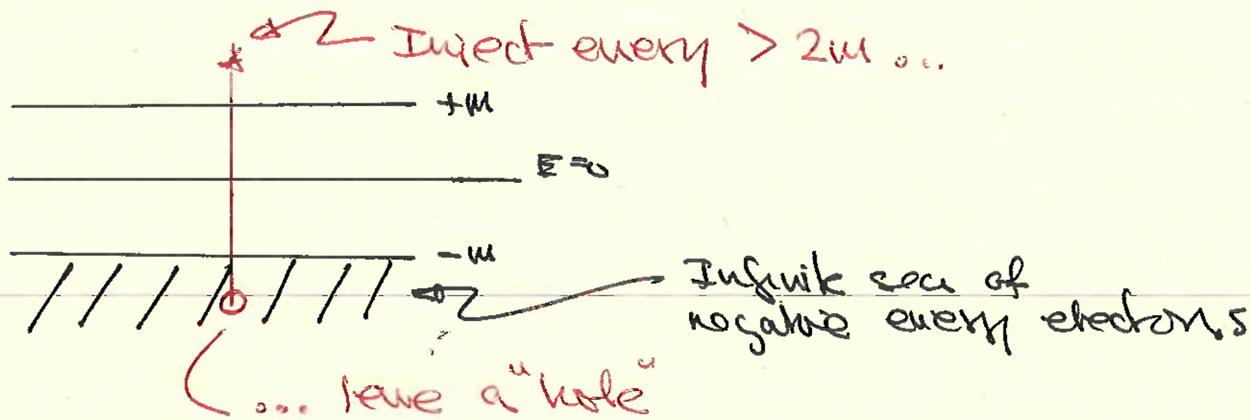
Also for $\vec{\Sigma} = \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$ Maybe I should write $\underline{\underline{\Sigma}}$

Have $\frac{\vec{\Sigma}}{2} \cdot \hat{p} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$ $\Rightarrow$ "1" and "3" are positive helicity
"2" and "4" are negative helicity

this smells a lot like spin-$1/2$!!

[Thursday: "Icing on the cake"]

Negative Energies (Dirac's Interpretation)



The "hole" should look like a positively charged electron

Anderson PR 43 (1933) 491 : Discovery of the positron

Cosmic Rays / Cloud Chamber