Phys 4702 Intro QM II Fall 2024 15 Oct 2024

* HW #7 Extension: Will Review Today
* Also do "parity" $\Rightarrow$ "Easy" HW #8

Review: Zeeman Effect in Hydrogen for $n=2$

Perturbation: $V = \frac{A}{2\hbar^2}(2\vec{L}\cdot\vec{S}) + \frac{B}{\hbar}(L_z + 2S_z)$

However but less intuitive $\Rightarrow$ $= \frac{A}{2\hbar^2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2) + \frac{B}{\hbar}(J_z + S_z)$

Deg. Subspace: $|n^{(0)}\rangle = |\ell m_\ell s m_s\rangle$ or $= |\ell s j m\rangle$ } Eight states either way!

Note: V = "spin orbit" + "Zeeman"

i.e. No V_K! $\Rightarrow$ Don't expect to get $\ell=0$ correct!

"Diagonalize Perturbation in Degenerate Subspace"

i.e. Find eigenvalues for 8×8 matrix $\langle m^{(0)}|V|n^{(0)}\rangle$

Using which basis? Not diagonal in either!!

But All terms diagonal in $|\ell s j m\rangle$ except S_z

"Handy"

i.e. $\langle \ell' s j' m'|(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)|\ell s j m\rangle = [j(j+1) - \ell(\ell+1) - \frac{1}{2}(\frac{1}{2}+1)]\hbar^2 \times \delta_{\ell'\ell}\delta_{j'j}\delta_{m'm}$

& $\langle \ell' s j' m'|J_z|\ell s j m\rangle = m\hbar\,\delta_{\ell'\ell}\delta_{j'j}\delta_{m'm}$

$\Rightarrow$ Just need to calculate $\langle \ell s j' m'|S_z|\ell s j m\rangle$!

How? write $|\ell s j m\rangle$ in terms of $|\ell m_\ell s m_s\rangle$!

i.e. $$|\ell s j m\rangle = \sum_{m_\ell, m_s} |\ell m_\ell s m_s\rangle \underline{\langle \ell m_\ell s m_s | \ell s j m\rangle}$$

"Clebsch Gordan Coefficients"

But $m_\ell + m_s = m$ (and $j = \ell \pm 1/2$)

↳ Just sum over m_s!

We Found

$$|j=\ell\pm\tfrac{1}{2}, m\rangle = \left[\frac{\ell \pm m + 1/2}{2\ell+1}\right]^{1/2} |\ell, m_\ell = m - \tfrac{1}{2}; +\tfrac{1}{2}\rangle \quad m_s = +\frac{1}{2}$$

$$\pm \left[\frac{\ell \mp m + 1/2}{2\ell+1}\right]^{1/2} |\ell, m_\ell = m + 1/2; -\tfrac{1}{2}\rangle \quad m_s = -\frac{1}{2}$$

Now just do the work!

Diagonal elements (Did this in class for weak field (no deg.))

$$\langle j=\ell\pm\tfrac{1}{2}, m | S_z | j=\ell\pm\tfrac{1}{2}, m\rangle = \left(\frac{\hbar}{2}\right)\frac{\ell \pm m + 1/2}{2\ell+1} + \left(\frac{-\hbar}{2}\right)\frac{\ell \mp m + 1/2}{2\ell+1}$$

$$= \frac{\hbar}{2}\frac{1}{2\ell+1}\left[\ell \pm m + \tfrac{1}{2} - (\ell \mp m + \tfrac{1}{2})\right] = \frac{m\hbar}{2\ell+1}$$

Off Diagonal elements

$$\langle j=\ell\mp\tfrac{1}{2}, m | S_z | j=\ell\pm\tfrac{1}{2}, m\rangle = \frac{\hbar}{2}\frac{-1}{2\ell+1}\left[(\ell - m + 1/2)(\ell + m + 1/2)\right]^{1/2}$$

$$- \frac{\hbar}{2}\frac{1}{2\ell+1}\left[(\ell + m \cdot 1/2)(\ell - m + 1/2)\right]^{1/2}$$

Check! $$= -\hbar \frac{1}{2\ell+1}\left[(\ell + 1/2)^2 - m^2\right]^{1/2}$$

i.e. the 8x8 matrix looks like

ℓ, s, j, m		1	2	3	4	5	6	7	8
$0, 1/2, 1/2, 1/2$	1	•							
$0, 1/2, 1/2, -1/2$	2		•						
$1, 1/2, 3/2, 3/2$	3			•					
$1, 1/2, 3/2, -3/2$	4				•				
$1, 1/2, 1/2, 1/2$	5					•	✓		
$1, 1/2, 3/2, 1/2$	6					✓	•		
$1, 1/2, 1/2, -1/2$	7							•	✓
$1, 1/2, 3/2, -1/2$	8							✓	•

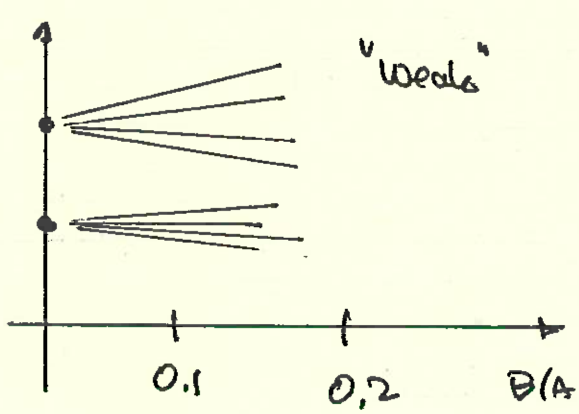
Just a pair of 2x2 matrices!

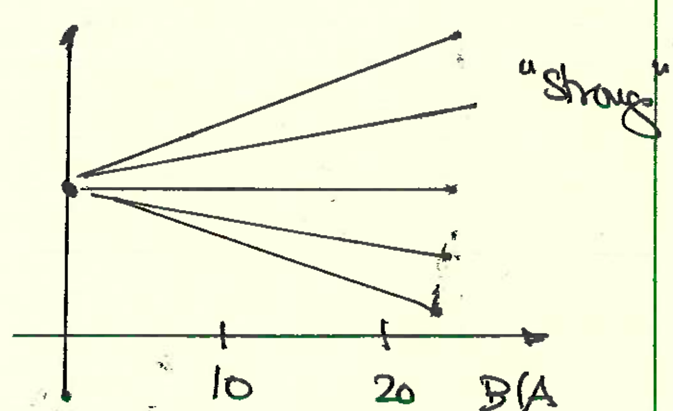
Now Finish it up

- "Diagonalize" to find the Δ's for mixed 5&6, 7&8
- Put together with "A" and "B"
- Plot as a function of B/A

Remember! Wrong for $\ell=0$ states at $B=0$!

& I artificially move them





Parity Symmetry

- we already saw this formally. ($\Rightarrow \mathbb{Z}_2$ group!)
- Let's go back to basics here.
- then revisit the formalism

One Dimensional Schrödinger Equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

Now consider $x \to y = -x$ "Reflection about origin"

$dy = -dx \quad dy^2 = dx^2$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dy^2} + V(-y)\psi(-y) = E\psi(-y)$$

i.e. $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(-x) + V(-x)\psi(-x) = E\psi(-x)$$

<u>Now Suppose</u> that $V(-x) = V(x)$ "Parity Symmetric"

$\Rightarrow \psi(-x)$ satisfies the same equation as $\psi(x)$!

i.e. $\psi(-x) = e^{i\alpha}\psi(x)$ But $\psi(-(-x)) = e^{i\alpha}\psi(-x)$

$$= e^{2i\alpha}\psi(x)$$

So $e^{2i\alpha} = 1 \Rightarrow \alpha = n\pi \quad n = \text{integer} \Rightarrow e^{i\alpha} = \pm 1$

$\Rightarrow \psi(-x) = \pm\psi(x)$ "Positive or Negative Parity"

Example: SHO in 1D 22 Feb and Concepts 3.5.2

$$V(x) = \frac{1}{2} m\omega^2 x^2 = V(-x)$$

$$\psi_n(x) = N \exp\left(-\frac{m\omega}{\hbar} x^2\right) \underline{H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right]}$$

"Hermite Polynomial"

Saw $H_n(x)$ has only even (odd) powers of x
for even (odd) values of $n = 0, 1, 2, 3, \ldots$

So $\psi_n(-x) = (-1)^n \psi_n(x)$ Even or odd parity!

HW: Infinite square well (w/ parity symmetry)
i.e. $-a \le x \le a$ NOT $0 \le x \le L$

Three Dimensional Schrödinger Equation

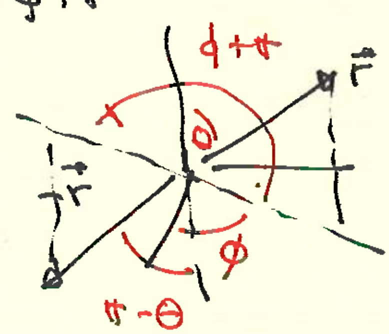
$$-\frac{\hbar^2}{2m} \vec{\nabla}^2 \psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) = E\psi(\vec{r})$$

Now "Reflection about origin" means $\vec{r} \to -\vec{r}$

Interesting Cases: Spherical Symmetry i.e. $V(\vec{r}) = V(r)$
i.e. $\psi(\vec{r}) = R_{n\ell}(r)\, Y_\ell(\theta, \phi)$

Now $\vec{r} \to -\vec{r}$ means $\theta \to \pi - \theta$ $\phi \to \phi + \pi$

Find (HW) $Y_\ell^m \to (-1)^\ell Y_\ell^m$



"Parity determined by ℓ"

Formalism (Again!)

Unitary symmetry operator $\mathcal{P}$

Must satisfy $\langle\alpha|\mathcal{P}^{+}\vec{r}\,\mathcal{P}|\alpha\rangle = -\langle\alpha|\vec{r}|\alpha\rangle$

$\Leftrightarrow \mathcal{P}^{+}\vec{r}\,\mathcal{P} = -\vec{r}$

But $\mathcal{P}\mathcal{P}^{+} = 1 \Rightarrow \vec{r}\,\mathcal{P} = -\mathcal{P}\vec{r}$ (see HW for implication)

Also $\mathcal{P}^2 = 1 \Rightarrow \mathcal{P} = \mathcal{P}^{+}$ Hermitian! Observable !!

Eigen values are ± 1

i.e. $\vec{r}\,\mathcal{P}|\vec{r}\,'\rangle = -\mathcal{P}\vec{r}\,|\vec{r}\,'\rangle = -\vec{r}\,'\,\mathcal{P}|\vec{r}\,'\rangle$

$\Rightarrow \mathcal{P}|\vec{r}\,'\rangle = |-\vec{r}\,'\rangle$ so $\mathcal{P}^2|\vec{r}\,'\rangle = |\vec{r}\,'\rangle$ etc...

Momentum

$T(d\vec{r}\,') = 1 - \frac{i}{\hbar}\vec{p}\cdot d\vec{r}\,'$

But $\mathcal{P}T(d\vec{r}\,') = T(-d\vec{r}\,')\mathcal{P}$

$\mathcal{P}T(d\vec{r}\,')|\vec{r}\,'\rangle$
$= \mathcal{P}|\vec{r}\,'+d\vec{r}\,'\rangle = |-\vec{r}-d\vec{r}\,'\rangle$
$= T(-d\vec{r}\,')|-\vec{r}\,'\rangle = T(-d\vec{r}\,')\mathcal{P}|\vec{r}\,'\rangle$

$\Leftrightarrow \mathcal{P} - \frac{i}{\hbar}\mathcal{P}\vec{p}\cdot d\vec{r}\,' = \mathcal{P} + \frac{i}{\hbar}\vec{p}\,\mathcal{P}\cdot d\vec{r}\,'$

$(-\mathcal{P}\vec{p})\cdot d\vec{r}\,' = (\vec{p}\,\mathcal{P})\cdot d\vec{r}\,' \Rightarrow \mathcal{P}^{(+)}\vec{p}\,\mathcal{P} = -\vec{p}$

However for Angular Momentum

$\mathcal{P}^{+}\vec{J}\,\mathcal{P} = +\vec{J}$

[Easy to see for $\vec{L}$,
also $\mathcal{D}(R) = 1 - \frac{i}{\hbar}\vec{J}\cdot\hat{u}\,\varepsilon$]

Lingo: $\vec{r}, \vec{p}$ are "polar" vectors

$\vec{J}$ is an "axial" vector