Phys 4702 Intro QM II Fall 2024 14 Nov 2024

* HW status

* Back to Quantum Mechanics

Relativity and Quantum Mechanics

[Recall: Natural Units! i.e. $\hbar = c = 1$]

Our Goal: Wave Mechanics consistent with Special Relativity.

- Will focus on free particle
- Will add EM interaction later via Gauge Symmetry

NOTE: "Doomed to fail!" Our postulates were based on probability!

$$\text{i.e. } \langle\alpha|\alpha\rangle = 1 = \int_{\text{all space}} \psi^*(\vec{r})\,\psi(\vec{r})\, d^3r$$

Reference Feshbach & Villars

But Relativity allows for "pair creation" etc.. What to do??

The Plan: Press forward, learn what we can

then fix problems w/ Quantum Fields (last class)

Back to Basics

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

* Natural Units!!

w/ $H|E_p\rangle = E_p|E_p\rangle$ E_p = Energy of Free particle w/ momentum p

But now $E_p = +[\vec{p}^{\,2} + m^2]^{1/2}$ the square root will be a source of trouble!

Follow your nose:

$$i\frac{\partial}{\partial t}\langle\vec{r}|\psi(t)\rangle = \langle\vec{r}|H|\psi(t)\rangle$$

$$= \int d^3p\,\langle\vec{r}|\vec{p}\rangle\langle\vec{p}|H|\psi(t)\rangle$$

$$= \int d^3p\,\frac{e^{i\vec{p}\cdot\vec{r}}}{(2\pi)^{3/2}}\,E_p\,\langle\vec{p}|\psi(t)\rangle$$

$$= \int d^3p\int d^3r'\,\frac{e^{i\vec{p}\cdot\vec{r}}}{(2\pi)^{3/2}}\,[p^2+m^2]^{1/2}\,\langle\vec{p}|\vec{r}'\rangle\langle\vec{r}'|\psi(t)\rangle$$

i.e. $$i\frac{\partial}{\partial t}\psi(\vec{r},t) = \int d^3r'\int d^3p\,\frac{e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}}{(2\pi)^3}\left[m+\frac{p^2}{2m}-\frac{p^4}{8m^3}+\cdots\right]\psi(\vec{r}',t)$$

ugly!! No way to make this covariant!

Klein-Gordon Equation

Different approach: Use the square of the Hamiltonian

$$i\frac{\partial}{\partial t}\left[i\frac{\partial}{\partial t}|\psi(t)\rangle\right] = H\left[H|\psi(t)\rangle\right]$$

$$-\frac{\partial^2}{\partial t^2}|\psi(t)\rangle = H^2|\psi(t)\rangle = [\vec{p}^{\,2}+m^2]|\psi(t)\rangle$$

Now do $\langle\vec{r}|$ on both sides and $\psi(\vec{r},t) = \langle\vec{r}|\psi(t)\rangle$

$$\Rightarrow -\frac{\partial^2}{\partial t^2}\psi(\vec{r},t) = \langle\vec{r}|[\vec{p}^{\,2}+m^2]|\psi(t)\rangle$$

$$= [-\cancel{\hbar}^2\vec{\nabla}^2+m^2]\,\psi(\vec{r},t)$$

i.e. $$\left[\frac{\partial^2}{\partial t^2}-\vec{\nabla}^2+m^2\right]\psi(\vec{r},t) = 0$$ "Klein-Gordon Eq."

Not so ugly. But it is also manifestly covariant!

Recall $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

i.e. $\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$

$\Rightarrow$ K.G. Eq is $\left[\partial_\mu \partial^\mu + m^2\right] \phi(x^\mu) = 0$

Solution $\phi(x^\mu) = N e^{-ip_\mu x^\mu} = N e^{-iE_p t} e^{i\vec{p}\cdot\vec{r}}$

where $\left[\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right] \phi(\vec{r},t) = \left[(-iE_p)^2 - (i\vec{p})^2 + m^2\right]\phi = 0$

---

i.e. $-E^2 + p^2 + m^2 = 0$

$\Rightarrow E = \pm\left[p^2 + m^2\right] = \pm E_p$

Negative Energy Solutions! (For a free particle!)

Need them! (Completeness!)

What do they mean? How can we use them??

Another Problem Consider the "conserved current"

$$j^\mu = \frac{i}{2m}\left[\phi^* \partial^\mu \phi - (\partial^\mu \phi)^* \phi\right]$$ Homework.

$\Rightarrow \rho(\vec{r},t) = j^0(\vec{r},t) = \frac{i}{2m}\left[\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t}\phi\right]$

is not positive-definite! What to do???

Hints by considering a <u>charged</u> particle:

$$\Longrightarrow E \to \underbrace{E - q\Phi} \quad \text{and} \quad \vec{p} \to \underbrace{\vec{p} - q\vec{A}}$$ Remember: $c = 1$

think of this as "adding $q\Phi$" to get eigenvalue

we got this from our rules for canonical momentum

i.e. $p^\mu = (E, \vec{p}) \to p^\mu - qA^\mu$ $\quad A^\mu = (\Phi, \vec{A})$

n.b. $A_\mu = (\Phi, -\vec{A})$

In Fact <u>A^μ is a four-vector!</u>

i.e. Transforms under a Lorentz Transformation.

<u>Olomian Book</u>: this is how electromagnetism is "built" as a covariant formulation.

"You just have to take my word for it."

How to incorporate this into

$$\left[\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right]\psi = [\partial_\mu \partial^\mu + m^2]\psi = 0 \quad ?$$

think about how we got here. [Sorry, not rigorous!]

$$p_\mu = (E, -\vec{p}) \quad \Rightarrow (E - q\Phi, -\vec{p} + q\vec{A}) = p_\mu - qA_\mu$$

$$\to (i\partial_t, i\vec{\nabla}) \quad \Rightarrow (i\partial_t - q\Phi, i\vec{\nabla} + q\vec{A})$$

$$= i(\partial_t + iq\Phi, \vec{\nabla} - iq\vec{A})$$

$$= i(\partial_\mu + iqA_\mu) \equiv iD_\mu$$

"Covariant Derivative"

$$\Longrightarrow \underline{\underline{[D_\mu D^\mu + m^2]\psi(x^\alpha) = 0}}$$

"Klein-Gordon Equation w/ Electromagnetism"

Note that $\psi^*(x^\alpha)$ satisfies

$$[D_\mu^* D^{\mu *} + m^2]\,\psi^*(x^\alpha) = 0$$

w/ $D_\mu^* = \partial_\mu - iqA_\mu = \partial_\mu + i(-q)A_\mu$

i.e. ψ^* satisfies K.G. for $q \Rightarrow -q$! Antiparticle??

- "Flailing away" before QFT settled in
- See Ref to Feshbach & Villars (posted)
- Last thing in this course: Klein - Gordon Field
- But now, a little more flailing

The Big Problem : No positive - definite probability density!

Next Week: Dirac's Brilliant Idea (+ Consequences)

Here's a way to work with K.G. ...

K.G. is second order in time $\Rightarrow$ Need two initial conditions

$\Leftarrow$ Try splitting $\psi(x^\alpha)$ into two pieces:

$$\phi(t,\vec{r}) = \frac{1}{2}\left[\psi(t,\vec{r}) + \frac{i}{m} D_t \psi(t,\vec{r})\right]$$ Giving up on Covariance!

$$\chi(t,\vec{r}) = \frac{1}{2}\left[\psi(t,\vec{r}) - \frac{i}{m} D_t \psi(t,\vec{r})\right]$$ [But now only need ϕ, χ @ $t=0$]

Watch: $iD_t\phi = \frac{1}{2}\left[iD_t\psi - \frac{1}{m} D_t^2 \psi\right]$ $[D_t^2\psi - \vec{D}^2\psi + m^2\psi = 0]$

$$= \frac{1}{2}\left[iD_t\psi - \frac{1}{m}\vec{D}^2\psi + m\psi\right]$$

$$= -\frac{1}{2m}\vec{D}^2\psi + \frac{1}{2}m\left[\psi + \frac{i}{m}D_t\psi\right]$$ $= \phi$

$= \phi + \chi$

$$\Rightarrow \quad iD_t\phi = -\frac{1}{2m}\vec{D}^2(\phi+\chi) + m\phi$$

$$\text{Also} \quad iD_t\chi = +\frac{1}{2m}\vec{D}^2(\phi+\chi) - m\chi$$

(*)

- Coupled 1st order in time PDE's
- "Schrödinger Eq." – like   Hmm...

So, what about the current?

$$j^\mu = \frac{i}{2m}\left[\psi^* D^\mu \psi - (D^\mu\psi)^*\psi\right]$$

$$\rho = j^0 = \frac{i}{2m}\left[\psi^* D_t \psi - (D_t\psi)^*\psi\right]$$

$$= \frac{1}{2m}\left[\psi^*(iD_t\psi) + (iD_t\psi)^*\psi\right]$$

$$= \frac{1}{2m}\left[(\phi+\chi)^*(iD_t\phi + iD_t\chi) + (iD_t\phi + iD_t\chi)^*(\phi+\chi)\right]$$

Substitute from (*)

$\Rightarrow$ the "$\vec{D}^2$" terms all cancel!

$$= \phi^*\phi - \chi^*\chi$$

Aha! Maybe interpretable? Charge!?

$\Rightarrow$ Alas, it never worked out.

Solution: QFT!

Next Weeks: Dirac's Approach $\Rightarrow$ Spin-1/2!!