Phys 4702 Intro QM II Fall 2024 12 Nov 2024

* HW Status
* Syllabus Change! Covariance, Rel. Unit today.

## Covariance in Special Relativity

Why?
- Need for Klein-Gordon & Dirac Equations!
- "Make your theories consistent with S.R.!"
- Preparation for study of General Relativity

Good References: Ohanian "Classical Electrodynamics, 2. Ed."

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## Vectors and Covectors

aka Contravariant Vectors and Covariant Vectors

aka "Sticks" and "Lasagnas"

See posted AJP article: "Questions & Answers"

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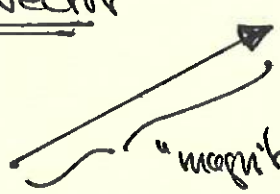
## Stick Vector



"Direction"

"magnitude" = length

Generic Notation: $\vec{A}$

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## Lasagna Vector

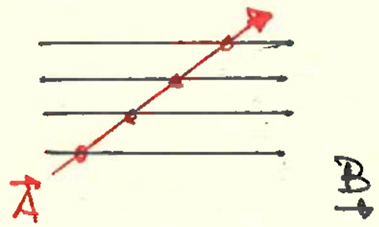
"Direction"

"magnitude" = density

Generic Notation: $\underset{\rightarrow}{A}$

Prototype: Gradient of Equipotential Surfaces.

**Inner Product:** "Put the stick in the lasagna"




$\vec{A} \cdot \underset{\rightarrow}{B}$ = # of times the stick pierces a lasagna noodle

NOTE "UP" AND "DOWN" ARROWS!

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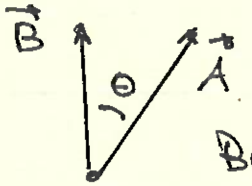
So far we have no way to define an angle!

i.e.




with $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$

But we can't do "$\vec{A} \cdot \vec{B}$"!

⇒ Must invent some way to turn $\underset{\rightarrow}{B}$ into $\vec{B}$!

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**Metric Tensor:** $\overset{\rightarrow\rightarrow}{g}$ and $\underset{\rightarrow\rightarrow}{g}$ with $\overset{\rightarrow\rightarrow}{g} \cdot \underset{\rightarrow\rightarrow}{g} = \overset{\rightarrow}{\underset{\rightarrow}{1}}$

i.e. $\vec{B} = \overset{\rightarrow\rightarrow}{g} \cdot \underset{\rightarrow}{B}$ and $\underset{\rightarrow}{A} = \underset{\rightarrow\rightarrow}{g} \cdot \vec{A}$

so $\underset{\rightarrow}{A} \cdot \vec{B} = (\vec{A} \cdot \underset{\rightarrow\rightarrow}{g}) \cdot (\overset{\rightarrow\rightarrow}{g} \cdot \underset{\rightarrow}{B}) = \vec{A} \cdot \underset{\rightarrow}{B}$

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**Towards 4D Spacetime:** $\vec{A} \Rightarrow A^\mu = (A^0, \vec{A})$   $\mu = 0, 1, 2, 3$

"Coordinate Representation"

"Position" $= x^\mu = (ct, \vec{r})$

Metric tensor $\overset{\rightarrow\rightarrow}{g} \Rightarrow g^{\mu\nu}$ w/ $A_\mu = g_{\mu\nu} A^\nu$ Implied Sum

General Relativity: $g^{\mu\nu} = g^{\mu\nu}(x^\alpha)$

Special Relativity: $g^{\mu\nu} = \eta^{\mu\nu} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

"Minkowski Metric"

NOTE: $\eta_{\mu\nu}$ is the same! i.e. $\eta_{\mu\alpha}\eta^{\alpha\nu} = \delta_\mu^\nu$

## Special Relativity: Constancy of the Speed of light

For all observers (i.e. observer #2 at V wrt #1)

$d\vec{r}^{\,2} = c^2 dt^2$ for an "expanding light front"

i.e. $ds^2 \equiv \underbrace{c^2dt^2 - d\vec{r}^{\,2}}_{\#1} = \underbrace{c^2dt'^2 - d\vec{r}'^{\,2}}_{\#2}$

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i.e. For position $x^\mu$ need $ds^2 = dx^\mu dx_\mu = dx^\mu g_{\mu\nu} dx^\nu$ to be "invariant"

$= \eta_{\mu\nu}$ from now on.

⇒ Define a four-vector $A^\mu$ as an object that transforms so that $A'^\mu A'_\mu = A^\mu A_\mu$ "Invariant"

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Lorentz Transformation (for a vector)

$A'^\mu = a^\mu{}_\nu A^\nu$ keeps inner product invariant!

Not hard to show that for observer #2 moving at speed V along the x-axis, ...

$$a^\mu{}_\nu = \begin{bmatrix} \frac{1}{\sqrt{1-V^2/c^2}} & \frac{-V/c}{\sqrt{1-V^2/c^2}} & 0 & 0 \\ \frac{-V/c}{\sqrt{1-V^2/c^2}} & \frac{1}{\sqrt{1-V^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

You should check! Don't want to take class time for the algebra!

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(for a covector)

$A'_\mu = \eta_{\mu\alpha} A'^\alpha = \eta_{\mu\alpha} a^\alpha{}_\beta A^\beta = \underline{\eta_{\mu\alpha} a^\alpha{}_\beta \eta^{\beta\nu}} A_\nu$

$= a_\mu{}^\nu A_\nu \Rightarrow a_\mu{}^\nu = \eta_{\mu\alpha} a^\alpha{}_\beta \eta^{\beta\nu}$ (HW)

## Important Four-Vectors

Position $x^\mu = (ct, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \vec{r})$

Velocity $u^\mu = dx^\mu / d\tau$ where $d\tau^2 = ds^2/c^2$

"Proper time" invariant!

$$= \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$$

$$= (c\beta, \beta v_x, \beta v_y, \beta v_z) \qquad \beta \equiv 1/\sqrt{1-v^2/c^2} \qquad v_i = \frac{dx^i}{dt}$$

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Momentum $p^\mu = m u^\mu$

NOTE • $u^\mu u_\mu = c^2$ Prove it!! $\Rightarrow p^\mu p_\mu = m^2c^2$

• $cp^0 = mc^2\beta = \dfrac{mc^2}{\sqrt{1-v^2/c^2}} = E$ "Energy"

$$\cong mc^2 + \frac{1}{2}mv^2 + \cdots$$

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Explicitly $p^\mu = (E/c, \vec{p}) \qquad \vec{p} = \beta m \vec{v}$

$p_\mu = (E/c, -\vec{p}) = \eta_{\mu\nu} p^\nu$

NOTE: Quantum Mechanical Plane Wave

$$\Psi(\vec{r}, t) = N e^{-iEt/\hbar} e^{i\vec{p}\cdot\vec{r}} = N e^{-ip_\mu x^\mu}$$ Covariant!!

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Why "covariance"?

- If an equation is "manifestly covariant" then it is trivially consistent with special rel.
- can build a theory based on all scalar inner products

## Calculus in 4D Spacetime

Four - Gradient $\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$

$\equiv \partial_\mu$ covector!! (HW!)

Also $\partial_\mu \partial^\mu A(x^\nu) = 0$ is the wave equation!

$\Rightarrow$ ON TO • Electrodynamics (not us!)

• Relativistic QM (start Thursday)

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## Natural Units ($\hbar = c = 1$)

• What's special about "speed of light"? Nothing!!

• We choose to measure $t$ in "seconds", but we might just as well measure $ct$ in "meters"

• Think of $c$ as a "conversion constant"

If time: "Parable of the surveyors"

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Little known fact: "c" has a defined value!

$c \equiv 299,792,458$ m/s (1983)

$\Rightarrow$ 1 meter $\equiv$ distance light travels in $1/c$ sec

[We can measure "time" very precisely.]

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So Drop the $c$ altogether and set $c \equiv 1$

[You can convert back later]

i.e. $mc^2 \to m$ in "energy units"

also momentum.

Typically we use MeV or GeV.

Example: Electron mass $m_e = 9.11 \times 10^{-31}$ kg

$\Leftrightarrow m_e c^2 = 8.19 \times 10^{-14}$ J $= 511 \times 10^3$ eV

$= \underline{\underline{0.511 \text{ MeV}}}$ Physicists know this, not this

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Do the same thing with $\hbar = 1$!

Recall $\Delta p \Delta x \sim \hbar = 1 \Rightarrow$ Distance in $\text{MeV}^{-1}$

$\Delta E \Delta t \sim \hbar = 1 \Rightarrow$ Time in $\text{MeV}^{-1}$

Useful: $\hbar c = 200 \text{ MeV} \cdot \text{fm} = 200 \times 10^6 \text{ eV} \times 10^{-15} \text{ m}$

$= 2 \times 10^{-7}$ eV·m

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Example: Ground State Energy of Hydrogen

From $2 = \left[\frac{2m_e}{-E}\right]^{1/2} \frac{e^2}{\hbar} \Rightarrow E = -\frac{1}{2} \frac{m_e e^4}{\hbar^2}$

We made this easier by using $\alpha \equiv \frac{e^2}{\hbar c} \simeq \frac{1}{137}$

i.e. $E = -\frac{1}{2} m_e c^2 \left(\frac{e^2}{\hbar c}\right)^2 = -\frac{1}{2}(0.511 \text{ MeV}) \left(\frac{1}{137}\right)^2 = -13.6 \text{ eV}$

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But we might have just said $E = -\frac{1}{2} m_e e^4 =$

$= -\frac{1}{2}(0.511 \text{ MeV}) \left(\frac{1}{137}\right)^2$

Thursday: Relativistic Quantum Mechanics!