Phys 4702 Intro QM II Fall 2024 8 Oct 2024

* Class over Zoom! (I'm in Boston.)
* Homework status

Review: Relativistic Corrections to Hydrogen Atom

Perturbation theory: $H = H_0 + V$

$$H_0 |n^{(0)}\rangle = E_n^{(0)} |n^{(0)}\rangle$$

$\Delta_n^{(1)} = \langle n^{(0)} | V | n^{(0)} \rangle$ in a basis for which V is Diagonal!

$\Rightarrow E_n = E_n^{(0)} + \Delta_n^{(1)} +$ Higher order corrections

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Hydrogen Atom: $H_0 = \frac{1}{2m}\vec{p}^{\,2} - \frac{Ze^2}{r}$ + "spin"

$|n^{(0)}\rangle = |n \ell m_\ell s m_s\rangle$ or $|n^{(0)}\rangle = |n \ell s j m\rangle$

$$E_n^{(0)} = -\frac{mZ^2e^4}{2\hbar^2 n^2} = -\frac{1}{2}mc^2 \frac{Z^2\alpha^2}{n^2} \qquad n = 1, 2, 3, \ldots$$

$\alpha \equiv e^2/\hbar c \cong 1/137$ Also $a_0 \equiv \hbar/mc\alpha \approx 0.5\,\text{Å}$

Relativistic Kinetic Energy

$$K = [\vec{p}^{\,2}c^2 + m^2c^4]^{1/2} - mc^2 = \frac{1}{2m}\vec{p}^{\,2} \underbrace{- \frac{1}{8}\frac{1}{m^3c^2}(\vec{p}^{\,2})^2}_{\equiv V_K} + \ldots$$

$$\Rightarrow \Delta_n^{(1)}\Big|_K = -\frac{1}{2}mc^2 Z^4\alpha^4 \left[-\frac{3}{4n^4} + \frac{1}{n^3(\ell+1/2)}\right]$$

Spin-Orbit Correction

$$V_{so} = -\vec{\mu}\cdot\vec{B} \qquad \vec{\mu} = -\frac{g}{2}\frac{e}{mc}\vec{S}$$

$$\vec{B} = -\frac{\vec{v}}{c}\times\vec{E} = \frac{\vec{v}}{c}\times\vec{\nabla}\Phi$$

Φ = Proton Electric Potential in Electron Rest Frame

Also $g \to g-1$ aka. $g \to \frac{1}{2}g$ since $g = 2$ (+ small)

$$\Rightarrow V_{so} = \frac{Ze^2}{2m^2c^2}\frac{1}{r^3}\vec{L}\cdot\vec{S}$$

We showed $2\vec{L}\cdot\vec{S}$ diagonal in $|\ell s j m\rangle$ basis

with $2\vec{L}\cdot\vec{S} = +\ell\hbar^2$ if $j = \ell + 1/2$

$= -(\ell+1)\hbar^2$ if $j = \ell - 1/2$

$$\Rightarrow \Delta_n^{(1)}\Big|_{so} = \frac{1}{4}mc^2\frac{Z^4\alpha^4}{n^3}\frac{1}{\ell(\ell+1)(\ell+1/2)}\begin{cases} +\ell & \text{if } j=\ell+1/2 \\ -(\ell+1) & \text{if } j=\ell-1/2 \end{cases}$$

<u>NOTE:</u> We could have also written

$$2\vec{L}\cdot\vec{S} = \vec{J}^2 - \vec{L}^2 - \vec{S}^2$$

i.e. Eigenvalues $\left[j(j+1) - \ell(\ell+1) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right]\hbar^2$

Puzzling Result for $l=0$:

For $l=0$ only $j=1/2$ is possible

$\Rightarrow$ the "l" is cancel!

- Good News: Infinity Avoided (or is it 0/0?)
- Bad News: $\Delta_n^{(1)}\Big|_{SO} \neq 0$ for $l=0$?

Today: Put the Pieces Together

$$\Delta_n^{(1)}\Big|_{SO;\, l=0} = \frac{1}{2} mc^2 \frac{Z^4\alpha^4}{n^3} \neq 0$$ How to address this?

$\Rightarrow$ "Darwin Term": See Townsend for discussion

Result from "non-relativistic reduction" of Dirac Equation:

$$V_D = -\frac{1}{8m^2c^2}\left[p_i, \left[p_i, V(r)\right]\right]$$

$\Rightarrow \Delta_n^{(1)}\Big|_D = $ Integral of $\delta(\vec{r}) \Rightarrow$ only s-states!

What's the physics? e^+e^- pair creation!

Breakdown distance $d \simeq c\Delta t = c\dfrac{\hbar}{mc^2} = \dfrac{\hbar}{mc}$

Note $d = \alpha a_0 \ll a_0$

$\Rightarrow$ Expect effects close to nucleus!
(i.e. s-states)

So let's add the two first order corrections:

$$\Delta_n^{(1)} = \Delta_n^{(1)}\Big|_k + \Delta_n^{(1)}\Big|_{SO}$$

Darwin term explains $\ell=0$!

$$= -\frac{1}{2}mc^2 Z^4\alpha^4 \left[-\frac{3}{4n^4} + \underline{\underline{\frac{1}{n^3(\ell+1/2)} - \frac{1}{2n^3}\frac{1}{\ell(\ell+1)(\ell+1/2)}}}\,(*) \begin{Bmatrix} +\ell \\ -(\ell+1) \end{Bmatrix} \right]$$

Do $j=\ell\pm 1/2$ separately (Leave out $1/n^3$ for now)

$$j=\ell+1/2: \quad \overset{(*)}{\frac{1}{\ell+1/2}} - \frac{1}{2(\ell+1)(\ell+1/2)} = \frac{2(\ell+1)-1}{2(\ell+1)(\ell+1/2)}$$

$$= \frac{2\ell+1}{2(\ell+1)(\ell+1/2)} = \frac{1}{\ell+1} = \frac{1}{j+1/2} \quad (**)$$

$$j=\ell-1/2: \quad \frac{1}{\ell+1/2} + \frac{1}{2\ell(\ell+1/2)} = \frac{2\ell+1}{2\ell(\ell+1/2)}$$

$$= \frac{1}{\ell} = \frac{1}{j+1/2} \quad (**)$$

SAME RESULT!

$$\Rightarrow \Delta_n^{(1)} = -\frac{1}{2}mc^2 Z^4\alpha^4 \left[-\frac{3}{4n^4} + \frac{1}{n^3}\frac{1}{j+1/2} \right]$$

Recall $\quad E_n^{(0)} = -\frac{1}{2}mc^2\frac{Z^2\alpha^2}{n^2}$

$$\Leftrightarrow E_n^{(0)} + \Delta_n^{(1)} \equiv E_{nj}$$

$$= -\frac{1}{2} mc^2 \frac{Z^2\alpha^2}{n^2} \left[1 + \frac{Z^2\alpha^2}{n^2} \left(\frac{n}{j+1/2} - \frac{3}{4} \right) \right]$$

"Hydrogen energy levels to first order in relativistic corrections"

<u>Notes</u>

- Energies depend on j (and n) <u>not</u> ℓ
  <u>ok</u>! $\vec{J}$ generates rotations, not $\vec{L}$!!
- "Relativistic Fine Structure" corrections are $\sim \alpha^2 \simeq 10^{-4}$

Recall "Hyperfine" interaction: $\vec{\mu}_e \cdot \vec{\mu}_p$

- Smaller by factor of $m_e/m_p \sim 1/2000$

[That's where the terminology comes from.]

<u>Homework</u>: Turn this into spectroscopy

i.e. $h\nu = \frac{hc}{\lambda} = \Delta E = E_{n'j'} - E_{nj}$

i.e. How do the "lines" split in wavelength?

Recall: $+\frac{1}{2} mc^2 \alpha^2 = 13.6 \text{ eV}$

Parting Comment

Obvious prediction: $2S_{1/2}$ and $2P_{1/2}$ have same energy!

1947: Lamb Shift : $\Delta E \cong 4 \times 10^{-6}$ eV

Predicted in Quantum Electrodynamics
(as is $g-2 = \alpha/2\pi + \cdots$)

Thursday : One more class on Hydrogen Atom

"Apply External Static $\vec{B} = B\hat{z}$ Field"

$\hookrightarrow$ Zeeman Effect!

Will have to go back and diagonalize!
Which basis?

Depends on "weak" or "strong" field.

i.e. Does $\vec{B}$ perturb the fine structure
or does the fine structure perturb $\vec{B}$?