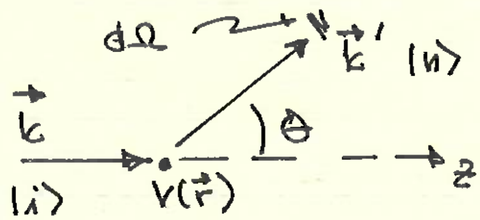
Phys 4702 Intro QM II Fall 2024 7 Nov 2024

* HW status

* Office Hrs Tomorrow: 'In and Out' (Sorry!)

Review: Elastic Scattering



$E = \hbar^2 k^2/2m$

$|\vec{k}| = |\vec{k}'|$ Lab = CM (massive target)

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{(mL^3)^2}{(2\pi\hbar)^2} |T_{ni}|^2$$

"T-Matrix"

$$T_{ni} = \langle n | V | \psi \rangle$$

$$|\psi\rangle = |i\rangle + \frac{1}{E - H_0 + i\varepsilon} V |\psi\rangle \qquad \underline{\varepsilon > 0}$$

"Lippman-Schwinger Equation"

NOTE: Could define operator T via $T|i\rangle = V|\psi\rangle$

$\Rightarrow$ Do "V" on Lippman-Schwinger to get

$$T = V + V \frac{1}{E - H_0 + i\varepsilon} T$$

$$= V + V \frac{1}{E - H_0 + i\varepsilon} V + V \frac{1}{E - H_0 + i\varepsilon} V \frac{1}{E - H_0 + i\varepsilon} T$$

etc.. $\Rightarrow$ "Perturbation Expansion"

Today: Scattering Amplitude

Find Equation for $\psi(\vec{r}) = \langle \vec{r} | \psi \rangle$ and "follow your nose".

$$\langle \vec{r}|\psi\rangle = \langle \vec{r}|i\rangle + \langle \vec{r}| \frac{1}{E-H_0+i\varepsilon} \overset{\overset{1}{\downarrow}}{} V|\psi\rangle$$

$$= \langle \vec{r}|i\rangle + \underline{\underline{\int d^3r'}}\ \underbrace{\langle \vec{r}| \frac{1}{E-H_0+i\varepsilon}|\vec{r}'\rangle}\ \underline{\underline{\overbrace{\langle \vec{r}'|V}^{V(\vec{r}')}}}\ \langle \vec{r}'|\psi\rangle \qquad (***)$$

Define the "<u>Scattering Green's Function</u>"

$$G(\vec{r},\vec{r}') \equiv \underline{\underline{\frac{\hbar^2}{2m}}}^{*} \langle \vec{r}| \overset{\downarrow 1}{} \frac{1}{E-H_0+i\varepsilon} \overset{\downarrow 1}{} |\vec{r}'\rangle$$

* For Dimensionality

$$= \frac{\hbar^2}{2m} \sum_{\vec{k}'} \sum_{\vec{k}''} \langle \vec{r}|\vec{k}'\rangle \langle \vec{k}'| \frac{1}{E-H_0+i\varepsilon} |\vec{k}''\rangle \langle \vec{k}''|\vec{r}'\rangle$$

$$\langle \vec{r}|\vec{k}'\rangle = \frac{1}{L^{3/2}} e^{i\vec{k}'\cdot\vec{r}} \qquad \langle \vec{k}''|\vec{r}'\rangle = \frac{1}{L^{3/2}} e^{-i\vec{k}''\cdot\vec{r}'}$$

<u>and</u> $\langle \vec{k}'| \dfrac{1}{E-H_0+i\varepsilon} |\vec{k}''\rangle = \dfrac{\delta_{\vec{k}';\vec{k}''}}{E-\hbar^2\vec{k}'^2/2m+i\varepsilon}$

However $E = \hbar^2k^2/2m$

$$\Leftrightarrow G(\vec{r},\vec{r}') = \frac{1}{L^3} \sum_{\vec{k}'} \frac{e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')}}{k^2-k'^2+i\varepsilon}$$

Redefined again, but still $\varepsilon > 0$!!

It is easy to let $L\to\infty$ and turn the sum into an integral!

$$k_i = \frac{2\pi}{L} n_i \Rightarrow \Delta k_i = \frac{2\pi}{L} = dk_i \Rightarrow d^3k = \frac{(2\pi)^3}{L^3}$$

4) $G(\vec{r},\vec{r}\,') = \frac{1}{(2\pi)^3}\int d^3k' \, \frac{e^{i\vec{k}'\cdot(\vec{r}-\vec{r}\,')}}{k^2-k'^2+i\varepsilon}$

Spherical coordinates: k', $\theta_{k'}$, $\phi_{k'}$

w/ angles relative to the $\vec{k}'$ direction.

i.e. $d^3k' = k'^2 \sin\theta_{k'} \, dk' \, d\phi_{k'}$

and $\vec{k}'\cdot(\vec{r}-\vec{r}\,') = k\,|\vec{r}-\vec{r}\,'|\cos\theta_k$

- Integral over $d\phi_k$ just gives 2π
- $\int_0^\pi \sin\theta_{k'}\, d\theta_{k'}\, e^{i\vec{k}'\cdot(\vec{r}-\vec{r}\,')} = \int_{-1}^{1} d(\cos\theta_k)\, e^{ik'|\vec{r}-\vec{r}\,'|\cos\theta_k}$

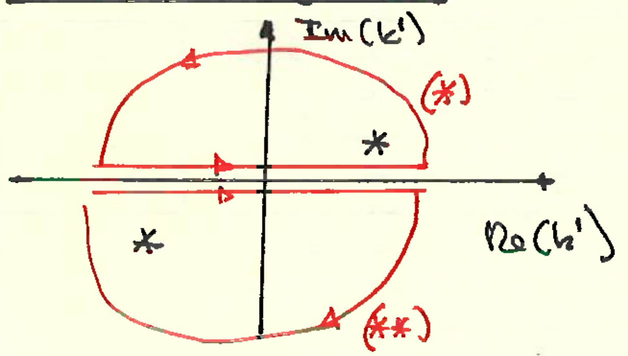
$$= \frac{1}{ik'|\vec{r}-\vec{r}\,'|}\left[e^{ik'|\vec{r}-\vec{r}\,'|} - e^{-ik'|\vec{r}-\vec{r}\,'|}\right]$$

$$G(\vec{r},\vec{r}\,') = \frac{1}{(2\pi)^2}\int_0^\infty k'dk' \, \frac{e^{ik'|\vec{r}-\vec{r}\,'|} - e^{-ik'|\vec{r}-\vec{r}\,'|}}{k^2-k'^2+i\varepsilon} \cdot \frac{1}{i|\vec{r}-\vec{r}\,'|}$$

EVEN!!

$$= \frac{1}{8\pi^2}\int_{-\infty}^{\infty} k'dk' \, \frac{\overset{(*)}{e^{ik'|\vec{r}-\vec{r}\,'|}} - \overset{(**)}{e^{-ik'|\vec{r}-\vec{r}\,'|}}}{k^2-k'^2+i\varepsilon} \cdot \frac{1}{i|\vec{r}-\vec{r}\,'|}$$

Contour Integration! Remember: $\varepsilon > 0$ Important!



Recall $\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$

Poles when $k'^2 - k^2 - i\varepsilon = 0$

i.e. $k' = \pm(k^2+i\varepsilon)^{1/2}$

$= \pm(k+i\varepsilon)$ $\quad * \equiv \pm k_0$

$$\int_{-\infty}^{\infty} k'dk' \, \frac{e^{ik'|\vec r-\vec r'|}}{-(k'-k_0^*)(k'+k_0)} \overset{(*)}{=} \oint \frac{-k' e^{ik'|\vec r - \vec r'|}/(k'+k_0)}{k'-k_0}\, dk'$$

$$= -2\pi i \, k_0 \, \frac{e^{ik_0|\vec r-\vec r'|}}{2k_0} = -\pi i \, e^{ik|\vec r-\vec r'|} \qquad \varepsilon \to 0$$

Similar for other integral : $k' = -k_0$ for pole
But clockwise contour $\Rightarrow$ extra minus sign
$\Rightarrow$ Same Result !!

$$\Rightarrow G(\vec r,\vec r') = \frac{1}{8\pi^2}\left[2\times(-\pi i)\, e^{ik|\vec r-\vec r'|}\right]\cdot\frac{1}{i|\vec r-\vec r'|}$$

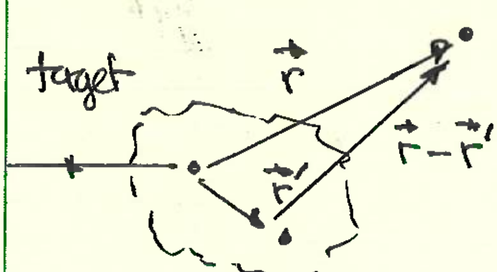
$$= -\frac{1}{4\pi}\,\frac{e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|}$$

Back to (***)

From Definition of $G(\vec r,\vec r')$

$$\underbrace{\langle \vec r|\psi\rangle}_{\psi(\vec r)} = \underbrace{\langle\vec r|i\rangle}_{\frac{1}{L^{3/2}}e^{i\vec k\cdot\vec r}} - \frac{2m}{\hbar^2}\int d^3r' \, \frac{e^{ik|\vec r-\vec r'|}}{4\pi|\vec r-\vec r'|}\, V(\vec r')\,\underbrace{\langle \vec r'|\psi\rangle}_{\psi(\vec r')}$$

Now consider detector far away from target:



$$|\vec r-\vec r'| = [(\vec r-\vec r')\cdot(\vec r-\vec r')]^{1/2}$$
$$\simeq [r^2 - 2\vec r\cdot\vec r']^{1/2}$$
$$\cong r - \hat r\cdot\vec r'$$

i.e. $e^{ik|\vec{r}-\vec{r}'|} \cong e^{ikr} e^{-ik\hat{r}\cdot\vec{r}'} = e^{ikr} e^{-i\vec{k}'\cdot\vec{r}'}$

Also take $|\vec{r}-\vec{r}'| = r$ in denominator

$$\Rightarrow \psi(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}} - \frac{2m}{4\pi\hbar^2} \frac{e^{ikr}}{r} \int d^3r' \, e^{-i\vec{k}'\cdot\vec{r}'} V(\vec{r}')\psi(\vec{r}')$$

outgoing spherical wave!

$$= \frac{1}{L^{3/2}}\left[e^{i\vec{k}\cdot\vec{r}} - \frac{mL^3}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int d^3r' \frac{e^{-i\vec{k}'\cdot\vec{r}'}}{L^{3/2}} V(\vec{r}')\psi(\vec{r}')\right.$$

$$= \frac{1}{L^{3/2}}\left[e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\vec{k},\vec{k}')\right]$$

w/ $f(\vec{k},\vec{k}') \equiv -\frac{mL^3}{2\pi\hbar^2} \int d^3r' \frac{e^{-i\vec{k}'\cdot\vec{r}'}}{L^{3/2}} V(\vec{r}')\psi(\vec{r}')$

"<u>Scattering Amplitude</u>" *

<u>NOTE</u>: $\langle \vec{r} | \vec{k}' \rangle = \frac{1}{L^{3/2}} e^{i\vec{k}'\cdot\vec{r}}$

$$\Rightarrow f(\vec{k},\vec{k}') = -\frac{mL^3}{2\pi\hbar^2} \int d^3r' \langle \vec{k}' | \vec{r}' \rangle V(\vec{r}') \langle \vec{r}' | \psi \rangle$$

$$= -\frac{mL^3}{2\pi\hbar^2} \langle \underset{\text{"}n\text{"}}{\vec{k}'} | V | \psi \rangle = -\frac{mL^3}{2\pi\hbar^2} T_{ni}$$

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i.e. $\frac{d\sigma}{d\Omega} = \frac{(mL^3)^2}{(2\pi\hbar^2)^2} |T_{ni}|^2 = |f(\vec{k},\vec{k}')|^2$

* Hence the name!

The Born Approximation: One of many approaches to calculating $f(\vec{k},\vec{k}')$!!

"Not a strong scatterer"

$$\Rightarrow \psi(\vec{r}) \cong \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$

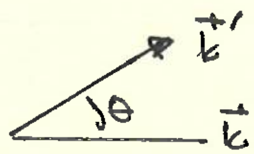
and $$f(\vec{k},\vec{k}') = -\frac{mL^3}{2\pi\hbar^2}\int d^3r' \frac{e^{-i\vec{k}'\cdot\vec{r}'}}{L^{3/2}} V(\vec{r}') \frac{e^{i\vec{k}\cdot\vec{r}'}}{L^{3/2}}$$

$(r' \to r)$

$$= -\frac{m}{2\pi\hbar^2}\int d^3r\, e^{i\vec{q}\cdot\vec{r}} V(\vec{r}) \qquad \vec{q} \equiv \vec{k} - \vec{k}'$$

"Momentum Transfer"

NOTE:

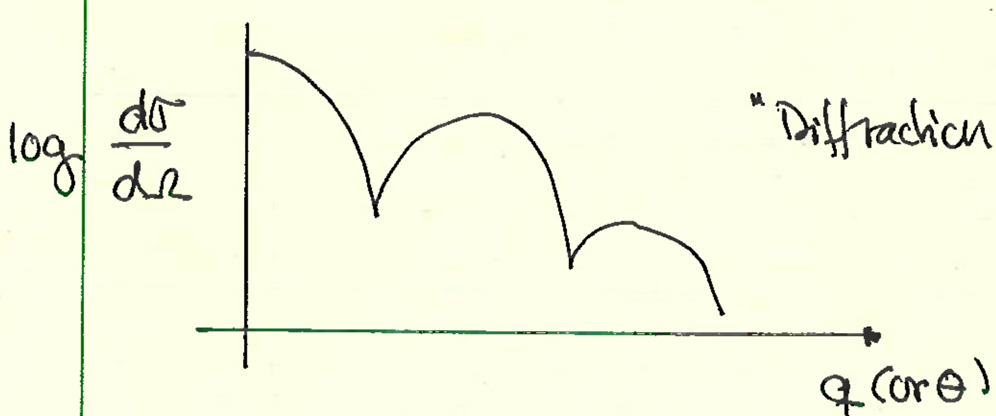


$$\vec{q}^{\,2} = \vec{k}^2 - 2\vec{k}\cdot\vec{k}' + \vec{k}'^2$$
$$= 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2}$$

Example: $V(r) = V_0$ for $r \le a$ $(=0$ for $r > a)$

$$\Rightarrow f(\theta) = -\frac{m}{2\pi\hbar^2} V_0 \underset{(d\phi)}{2\pi} \int_0^a r^2 dr \int_{-1}^{1} e^{iqr\cos\theta}\, d(\cos\theta)$$

$$= -\frac{2m}{\hbar^2}\frac{V_0 a^3}{(qa)^2}\left[\frac{\sin(qa)}{qa} - \cos(qa)\right]$$



"Diffraction Pattern"