Phys 4702 Intro QM II   Fall 2024   5 Dec 2024

* Last class!

* HW #13 on Tuesday; Office Hours Tomorrow

Review: Klein-Gordon Equation

$$\left[\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right]\psi(\vec{x},t) = 0^{(*)} \quad \text{(Free Particle!)}$$

Problems: • Need negative energies $E = -E_p = -(p^2+m^2)^{1/2}$

• No clear interpretation of "current"

↳ It was discarded in favor of Dirac Equation

Review: Second Quantization (aka QFT)

Single Particle States $|\alpha_i\rangle \Rightarrow |\psi\rangle = |n_1, n_2, \ldots, n_i, \ldots\rangle$

"Creation and Annihilation Operators"

Bosons: $[a_i^+, a_j^+] = 0 = [a_i, a_j]$ & $\underline{[a_i, a_j^+] = \delta_{ij}}$

Fermions: $\{a_i^+, a_j^+\} = 0 = \{a_i, a_j\}$ & $\underline{\{a_i, a_j^+\} = \delta_{ij}}$

Forgot to be explicit on Tuesday!

Today: The Klein-Gordon Field

Approach: Find Hermitian Field operator $\Phi(\vec{r},t)$ to (*)

and consider $\psi(\vec{r},t)$ as some linear combination

↳ Write $\Phi(\vec{r},t) = \frac{1}{L^{3/2}} \sum_{\vec{k}} q_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{r}}$

Note: $\underline{q_{\vec{k}}^+ = q_{-\vec{k}}}$ (*)

"Big Box" normalization

↳ the $\vec{k}$ are discrete

Plug in:

$$\frac{1}{L^{3/2}} \sum_{\vec{k}} \left[\ddot{q}_{\vec{k}} + \vec{k}^2 q_{\vec{k}} + m^2 q_{\vec{k}} \right] e^{i\vec{k}\cdot\vec{r}} = 0$$

$$\Leftrightarrow \ddot{q}_{\vec{k}} + \omega_{\vec{k}}^2 q_{\vec{k}} = 0 \qquad \omega_{\vec{k}}^2 \equiv (\vec{k}^2 + m^2) = \underline{\underline{\omega_{-\vec{k}}^2}}$$

It sure looks like a harmonic oscillator!
(But it's actually lots of harmonic oscillators.)

Note: $q_{\vec{k}}^+$ is not Hermitian! How to deal with it?
Recall: Two independent SHO's...

$$x = \frac{1}{(2m\omega)^{1/2}} (a_x + a_x^+) \qquad y = \frac{1}{(2m\omega)^{1/2}} (a_y + a_y^+)$$

Define $q_\pm \equiv \left(\frac{m}{2}\right)^{1/2} (x \pm iy) \Rightarrow \underline{\underline{q_+^+ = q_-}}$ (*)

$$\Leftrightarrow q_+ = \frac{1}{(2\omega)^{1/2}} (a_+ + a_-^+) \qquad a_\pm \equiv \frac{1}{\sqrt{2}} (a_x \pm i a_y)$$

Easy to show: $[a_+, a_+^+] = 1 = [a_-, a_-^+]$
and $[a_+, a_-^+] = 0 = [a_+, a_+^*] = [a_+^+, a_-^+]$
" $a_\pm$ are operators of two independent oscillators "

This tells us how to write the $q_{\vec{k}}(t)$ in terms of creation and annihilation operators!

$$q_{\vec{k}}(t) = \frac{1}{(2\omega_{\vec{k}})^{1/2}} \left[a_{\vec{k}} e^{-i\omega_{\vec{k}} t} + a_{-\vec{k}}^+ e^{i\omega_{\vec{k}} t} \right]$$

$$\Leftrightarrow \vec{q}_{\vec{k}}^{\,+}(t) = \vec{q}_{-\vec{k}}(t) \quad \text{with} \quad [a_{\vec{k}}, a_{\vec{k}'}^+] = \delta_{\vec{k},\vec{k}'}$$

<u>NOTE</u>: the Klein-Gordon quanta are Bosons!
No other degrees of freedom
$\Rightarrow$ Applicable to Spin-zero Particles!
e.g. Pionic Atoms (See MQM3e Problem 8.7)

Ok, we've concocted a Hermitian field operator that solves the Klein-Gordon Equation:

$$\Phi(\vec{r},t) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \frac{1}{(2\omega_{\vec{k}})^{1/2}} \left[a_{\vec{k}} e^{-i\omega_{\vec{k}} t} + a^{+}_{-\vec{k}} e^{i\omega_{\vec{k}} t} \right] e^{i\vec{k}\cdot\vec{r}}$$

But Does this make sense?
$\Rightarrow$ Let's consider some observables and see!

<u>Hamiltonian</u>: this one is easy!

Each oscillator contributes $(n_{\vec{k}} + \frac{1}{2})(\hbar)\omega_{\vec{k}} = (a^{+}_{\vec{k}} a_{\vec{k}} + \frac{1}{2})\omega_{\vec{k}}$

$$\Rightarrow H = \sum_{\vec{k}} (a^{+}_{\vec{k}} a_{\vec{k}} + \tfrac{1}{2}) \cancel{\hbar}\omega_{\vec{k}} \qquad \omega_{\vec{k}} = (\hbar^2 \vec{k}^2 + m^2)^{1/2} = E_p$$

$$= \sum_{\vec{k}} n_{\vec{k}} \omega_{\vec{k}} + E_0 \qquad E_0 \equiv \sum_{\vec{k}} \tfrac{1}{2}\omega_{\vec{k}} \text{ (infinite!)}$$

But it is Positive Definite! No more neg. energy!

<u>Momentum</u>: this is a little trickier

we expect $\vec{P} = \sum_{\vec{k}} \vec{k}\, a^{+}_{\vec{k}} a_{\vec{k}}$ but does this have the right properties

e.g. <u>(Recall)</u> $[\vec{P}, F(\vec{r})] = -i\vec{\nabla}F$

Check this out with $F(\vec{r}) = \Phi(\vec{r},t)$!

$$[\vec{P}, \Phi(\vec{r},t)] = \sum_{\vec{k}} \sum_{\vec{k}'}$$

$$\left[\vec{k}\, a^+_{\vec{k}} a_{\vec{k}} \,,\, \frac{1}{L^{3/2}} \frac{1}{(2\omega_{\vec{k}'})^{1/2}} \left(a_{\vec{k}'} e^{-i\omega_{\vec{k}'} t} + a^+_{-\vec{k}'} e^{i\omega_{\vec{k}'} t}\right) e^{i\vec{k}'\cdot\vec{r}}\right]$$

Need those commutators

$$[a^+_{\vec{k}} a_{\vec{k}}, a_{\vec{k}'}] = a^+_{\vec{k}} a_{\vec{k}} a_{\vec{k}'} - a_{\vec{k}'} a^+_{\vec{k}} a_{\vec{k}}$$
$$= [a^+_{\vec{k}}, a_{\vec{k}'}]\, a_{\vec{k}} = -\delta_{\vec{k},\vec{k}'}\, a_{\vec{k}}$$

$$[a^+_{\vec{k}} a_{\vec{k}}, a^+_{-\vec{k}'}] = a^+_{\vec{k}} a_{\vec{k}} a^+_{-\vec{k}'} - a^+_{-\vec{k}'} a^+_{\vec{k}} a_{\vec{k}}$$
$$= a^+_{\vec{k}} [a_{\vec{k}}, a^+_{-\vec{k}'}] = +\delta_{\vec{k},-\vec{k}'}\, a^+_{\vec{k}}$$

$$\Rightarrow [\vec{P}, \Phi(\vec{r},t)] = \frac{1}{L^{3/2}} \sum_{\vec{k}} \sum_{\vec{k}'} \frac{1}{(2\omega_{\vec{k}'})^{1/2}}\, \vec{k}$$
$$\times \left[-\delta_{\vec{k},\vec{k}'} a_{\vec{k}} e^{-i\omega_{\vec{k}'} t} + \delta_{\vec{k},-\vec{k}'} a^+_{\vec{k}} e^{i\omega_{\vec{k}'} t}\right] e^{i\vec{k}'\cdot\vec{r}}$$

$$\cancel{= \frac{1}{L^{3/2}} \sum_{\vec{k}} \frac{1}{(2\omega_{\vec{k}})^{1/2}} \left[-\vec{k} a_{\vec{k}} e^{-i\omega_{\vec{k}} t} + (-\vec{k}\right.}$$

$$= \frac{1}{L^{3/2}} \sum_{\vec{k}} \frac{1}{(2\omega_{\vec{k}})^{1/2}}\, \vec{k} \left[-a_{\vec{k}} e^{-i\omega_{\vec{k}} t} e^{i\vec{k}\cdot\vec{r}} + a^+_{\vec{k}} e^{i\omega_{\vec{k}} t} e^{-i\vec{k}\cdot\vec{r}} \right]$$

$$= -i\vec{\nabla} \left\{ \frac{1}{L^{3/2}} \sum_{\vec{k}} \frac{1}{(2\omega_{\vec{k}})^{1/2}} \left[-a_{\vec{k}} e^{-i\omega_{\vec{k}} t} e^{i\vec{k}\cdot\vec{r}} - a^+_{\vec{k}} e^{i\omega_{\vec{k}} t} e^{-i\vec{k}\cdot\vec{r}} \right] \right.$$

$\vec{k} \to -\vec{k}$

$$= i\vec{\nabla} \left\{ \frac{1}{L^{3/2}} \sum_{\vec{k}} \frac{1}{(2\omega_{\vec{k}})^{1/2}} \left[a_{\vec{k}} e^{-i\omega_{\vec{k}} t} + a^+_{-\vec{k}} e^{i\omega_{\vec{k}} t} \right] e^{i\vec{k}\cdot\vec{r}} \right.$$

i.e. $[\vec{P}, \Phi(\vec{r},t)] = i\vec{\nabla}\Phi(\vec{r},t)$ (Agrees w/ Huang 2.31)

Almost! Did I drop a minus sign somewhere??

Probability Current "the Biggie"

Consider a complex solution $\varphi(\vec{r},t)$ formed from two independent fields $\Phi_1(\vec{r},t)$ and $\Phi_2(\vec{r},t)$

i.e. $$\Phi_j(\vec{r},t) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \frac{1}{(2\omega_{\vec{k}})^{1/2}} \left[a_{j\vec{k}} e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)} + a^{\dagger}_{j\vec{k}} e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)} \right]$$

Did $\vec{k} \to -\vec{k}$

w/ $[a_{j\vec{k}}, a^{\dagger}_{j'\vec{k}'}] = \delta_{jj'}\delta_{\vec{k}\vec{k}'}$ and $[a_{j\vec{k}}, a_{j'\vec{k}'}] = 0$

$$\varphi(\vec{r},t) = \frac{1}{\sqrt{2}}\left[\phi_1(\vec{r},t) + i\phi_2(\vec{r},t)\right]$$ (NON HERMITIAN!)

$$= \frac{1}{L^{3/2}} \sum_{\vec{k}} \frac{1}{(2\omega_{\vec{k}})^{1/2}} \left[b_{\vec{k}} e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)} + c_{\vec{k}} e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)} \right]$$

w/ $b_{\vec{k}} \equiv \frac{1}{\sqrt{2}}(a_{1\vec{k}} + i a_{2\vec{k}})$ $\quad c_{\vec{k}} \equiv \frac{1}{\sqrt{2}}(a_{1\vec{k}} - i a_{2\vec{k}})$

Easy to show: $[b_{\vec{k}}, b^{\dagger}_{\vec{k}}] = 1 = [c_{\vec{k}}, c^{\dagger}_{\vec{k}}]$

$[b_{\vec{k}}, c^{\dagger}_{\vec{k}}] = 0$ "Independent Fields"

and $b^{\dagger}_{\vec{k}} b_{\vec{k}} + c^{\dagger}_{\vec{k}} c_{\vec{k}} = a^{\dagger}_{1\vec{k}} a_{1\vec{k}} + a^{\dagger}_{2\vec{k}} a_{2\vec{k}}$

"Maintains the counting"

⇒ $H = \sum_{\vec{k}} (b^+_{\vec{k}} b_{\vec{k}} + c^+_{\vec{k}} c_{\vec{k}})\, \omega_{\vec{k}} + \underline{\underline{\sum_{\vec{k}} \omega_{\vec{k}}}}$ "E_0"

$$\vec{P} = \sum_{\vec{k}} (b^+_{\vec{k}} b_{\vec{k}} + c^+_{\vec{k}} c_{\vec{k}})\, \vec{k}$$

Recall "conserved current"

$$j^\mu = c \times [\psi(\partial^\mu \psi^+) - \psi^+ (\partial^\mu \psi)]$$

⇒ Conserved <u>charge</u> $= c \int_{L^3} \left[\psi \frac{\partial \psi^+}{\partial t} - \psi^+ \frac{\partial \psi}{\partial t} \right] d^3 r$

⇒ <u>Find</u> (HW) $Q = ic \sum_{\vec{k}} [\underbrace{b^+_{\vec{k}} b_{\vec{k}}}_{\text{positive}} - \underbrace{c^+_{\vec{k}} c_{\vec{k}}}_{\text{negative}}]$

"The fields $b^+_{\vec{k}}$ and $c^+_{\vec{k}}$ create particles and antiparticles, respectively."

The End