Phys 4702 Intro QM II   Fall 2024   3 Dec 2024

* Last week of classes!

SFF's!!

* HW #13: – Maybe some changes
  – You can collaborate
  – I will grade it

Today: Identical Particles

aka Intro to "Quantum Field Theory"

Fundamental Problem! Deserves much more time than we can devote to it.

"Fermions" vs "Bosons": Relativistic Quantum Fields!

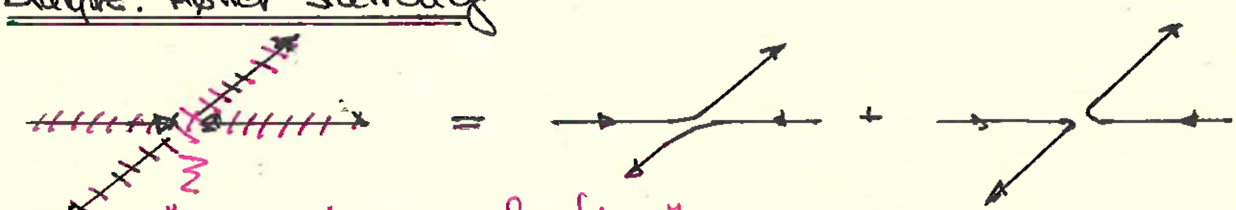
"half integer spin" e.g. Spin-$1/2$ "Electrons"

"integer spin" e.g. spin-0, spin-1 "prototype" "photons"

Example: Helium Atom

two electrons. Completely overlapping probability distributions. Can't tell which is which!

Example: Møller Scattering



"overlapping wave functions"

⇒ Must add amplitudes!

We need a formalism to deal with this!

nb. There are choices for the formalism!

Permutation Symmetry : Warning – the Notation is tricky

Let $|a\rangle$ and $|b\rangle$ be "single particle states"
[aka "orbitals" to chemists]

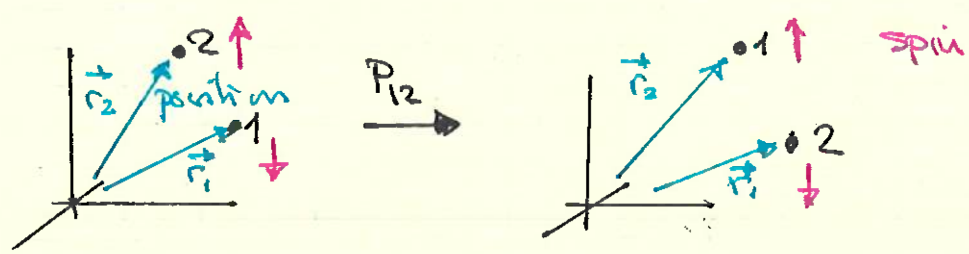
$\Leftrightarrow\ |a,b\rangle = |a\rangle \otimes |b\rangle = |a\rangle|b\rangle = |a\rangle_1 |b\rangle_2$

is a two particle state with #1 in $|a\rangle$ and #2 in $|b\rangle$

Exchange operator P_{12}

$P_{12}|a,b\rangle = |b,a\rangle$ i.e. $P_{12}\left(|a\rangle_1 |b\rangle_2\right) = |b\rangle_1 |a\rangle_2$

i.e.



Let $|\psi\rangle$ be an arbitrary two-particle state

Must have $P_{12}|\psi\rangle = e^{i\delta}|\psi\rangle$ so $|\psi\rangle$ is physical!

Also $P_{12}^2|\psi\rangle = e^{2i\delta}|\psi\rangle = |\psi\rangle$ Caveat: Anyons in 2D !!

$\Leftrightarrow\ 2i\delta = 2\pi i \times n \Rightarrow \delta = n\pi \Rightarrow P_{12}|\psi\rangle = \pm |\psi\rangle$

"Exchange Degeneracy"

$|\psi_S\rangle = \frac{1}{\sqrt{2}}|a,b\rangle + \frac{1}{\sqrt{2}}|b,a\rangle \Rightarrow P_{12}|\psi_S\rangle = +|\psi_S\rangle$

$|\psi_A\rangle = \frac{1}{\sqrt{2}}|a,b\rangle - \frac{1}{\sqrt{2}}|b,a\rangle \Rightarrow P_{12}|\psi_A\rangle = -|\psi_A\rangle$

Must be one or the other. But which one??

Nature to the rescue: Bosons and Fermions

nb. Need Relativistic Quantum Field theory to "Prove"!!

Spin $s = 0, 1, 2, \ldots \Rightarrow |\psi\rangle = |\psi_S\rangle$ "Bosons"

$s = 1/2, 3/2, 5/2, \ldots \Rightarrow |\psi\rangle = |\psi_A\rangle$ "Fermions"

Note: $|a\rangle = |b\rangle \Rightarrow |\psi_A\rangle = 0$ "Pauli Exclusion Principle"

NOT TRUE FOR BOSONS! $\Rightarrow$ "Superfluid Helium"

"Bose-Einstein Condensate"

Helium Atom: Good Problem to Work through!

$$H = \frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} - \frac{Ze^2}{r_1} - \frac{Ze^2}{r_2} + \underbrace{\frac{e^2}{|\vec{r}_1 - \vec{r}_2|}}_{\text{"the Hard Part"}}$$

- Pert theory, Variational MIQM2e Sec 12.2, MQM3e Sec 7.4
- Density Functional theory MQM3e Sec 7.6

Two Fermions*: Same Spatial State *Take spin-1/2

Let $|\alpha\rangle = |n; +\hat{z}\rangle = |n\rangle|+\hat{z}\rangle = |n;+\rangle = |n\rangle|+\rangle$

$|\beta\rangle = |n; -\hat{z}\rangle = |n;-\rangle$ etc...

We carelessly say "one spin up the other down"

but this is inaccurate!

Must have $|\psi\rangle = |\psi_A\rangle = \frac{1}{\sqrt{2}}\left[|\alpha\rangle_1 |\beta\rangle_2 - |\beta\rangle_1 |\alpha\rangle_2 \right]$

$$= |n\rangle \frac{1}{\sqrt{2}} \left[|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right]$$

Recall: Total spin = 0 state!!

Two Fermions: Arbitrary single particle states

"Slater Determinant" * 2! = 2 permutations

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} |\alpha\rangle_1 & |\beta\rangle_1 \\ |\alpha\rangle_2 & |\beta\rangle_2 \end{vmatrix} = \frac{1}{\sqrt{2}}\left[|\alpha\rangle_1 |\beta\rangle_2 - |\beta\rangle_1 |\alpha\rangle_2 \right]$$

orthogonal!

NOTE: P_{12} reverses rows $\Rightarrow$ (−) sign is automatic!
Also $|\beta\rangle = |\alpha\rangle \Rightarrow$ Two identical columns $\Rightarrow$ Zero!

N Fermions: Obvious Generalization

$$|\psi\rangle = \frac{1}{\sqrt{N!}} \begin{vmatrix} |\alpha\rangle_1 & |\beta\rangle_1 & |\gamma\rangle_1 & \cdots \\ |\alpha\rangle_2 & |\beta\rangle_2 & |\gamma\rangle_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ |\alpha\rangle_N & |\beta\rangle_N & |\gamma\rangle_N & \cdots \end{vmatrix} \Rightarrow \text{Atoms!!}$$

Second Quantization (i.e. Quantum Fields)

A Different Approach to multiparticle quantum mechanics

"The Bible": Quantum Theory of Many Particle Systems
Fetter & Walecka McGraw-Hill (1971) ⇒ Dover
... and the "battle" of Bjorken & Drell.

Fock Space

Given eigenstates $|\alpha_1\rangle, |\alpha_2\rangle \ldots |\alpha_i\rangle, \ldots$

$\Leftrightarrow |\psi\rangle = |n_1, n_2, \ldots, n_i, \ldots\rangle$ n_i = # of particles in $|\alpha_i\rangle$

$|0, 0, \ldots, 0, \ldots\rangle = |0\rangle$ "Vacuum"

$|0, 0, \ldots, 1, \ldots\rangle = |\alpha_i\rangle$ "Single particle state"

"Field operator" $a_i^+ |n_1, n_2, \ldots, n_i, \ldots\rangle \propto |n_1, n_2, \ldots, n_i+1, \ldots\rangle$

proportionality const TBD

Postulate that $a_i^+|0\rangle = |\alpha_i\rangle$

$\Rightarrow 1 = \langle\alpha_i|\alpha_i\rangle = [\langle 0|a_i][a_i^+|0\rangle] = \langle 0|a_i|\alpha_i\rangle$

i.e. Need $a_i|\alpha_i\rangle = |0\rangle$

"a_i^+ and a_i behave like creation and annihilation"

Also postulate that $a_i|\alpha_j\rangle = 0$ if $i \neq j$

$\Rightarrow a_i|\alpha_j\rangle = \delta_{ij}|0\rangle$

Now $a_j^+ a_i^+|0\rangle$ means "#1 in i, #2 in j"

$\Rightarrow a_i^+ a_j^+|0\rangle = \pm a_j^+ a_i^+|0\rangle$ for Bosons / Fermions

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Generalize to multiparticle states:

$a_i^+ a_j^+ - a_j^+ a_i^+ = 0 = [a_i^+, a_j^+]$ Bosons

$a_i^+ a_j^+ + a_j^+ a_i^+ = 0 = \{a_i^+, a_j^+\}$ Fermions

Adjoint: $[a_i, a_j] = 0$ Bosons $\{a_i, a_j^+\} = 0$ Fermions

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Number Operator $N_i = a_i^+ a_i$

$\Rightarrow N = \sum_i N_i = \sum_i a_i^+ a_i$

$\Rightarrow$ Easy to make an "additive" single particle operator into a field operator!

eq. k_i = Kinetic energy in state $|\alpha_i\rangle$

$\Rightarrow K = \sum_i k_i n_i = \sum_i k_i a_i^+ a_i$

$\Rightarrow$ Different bases, two-body operators, etc...

Applications of Second Quantization

F&W Chap. 1

1) Degenerate Electron Gas (MQM 3e Sec. 7.7.3)

$$H = H_{el} + H_b + H_{el-b}$$ ← electrons w/ positive background

$$\sum_i \frac{\vec{P}_i^2}{2m} + \sum_{i,j} \frac{e^2}{|\vec{r}_i - \vec{r}_j|}$$

"positive background" ⇒ uniform

$\boxed{\text{FERMIONS!}}$

Convert to momentum basis in second quantization
⇒ Good model for an alkali metal like sodium!

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2) Electromagnetic Field (MQM 3e Sec 7.8) $\boxed{\text{Bosons!}}$

"Big Box" solution of Maxwell's Equations

$$\vec{A}(\vec{r},t) = \sum_{\vec{k},\lambda}' \hat{e}_{\vec{k},\lambda} \underline{A_{\vec{k},\lambda}(\vec{r},t)}$$

Individual solutions to wave equation

Make proportional to Quantum Fields!

$$\Rightarrow H = \sum_{\vec{k},\lambda}' \hbar\omega_k a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + E_0$$ ← "vacuum energy"

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3) Klein-Gordon Field (MQM 3e Sec. 8.1.5)

Thursday!
"The Negative Energy Problem Vanishes"