PHYS4702 Intro Quantum Mechanics II HW#13 Due 10 Dec 2024

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Form the matrices $\underline{\underline{\vec{\gamma}}} = \underline{\underline{\gamma}}^0 \underline{\underline{\vec{\alpha}}}$ where $\underline{\underline{\gamma}}^0 = \underline{\underline{\beta}}$ and $\underline{\underline{\vec{\alpha}}}$ and $\underline{\underline{\beta}}$ were given in class. Then show that $(\underline{\underline{\gamma}}^0)^2 = \underline{\underline{1}}$, $(\underline{\underline{\gamma}}^i)^2 = -\underline{\underline{1}}$, $\underline{\underline{\gamma}}^0 \underline{\underline{\gamma}}^i = -\underline{\underline{\gamma}}^i \underline{\underline{\gamma}}^0$, and $\underline{\underline{\gamma}}^i \underline{\underline{\gamma}}^j = -\underline{\underline{\gamma}}^j \underline{\underline{\gamma}}^i$ for $i, j = 1, 2, 3$. Why did we require these conditions on the γ^μ be met?

(2) The Dirac Hamiltonian for a free particle is $\underline{\underline{H}} = \underline{\underline{\vec{\alpha}}} \cdot \vec{p} + \underline{\underline{\beta}} m$. Find the (four) energy eigenvalues E and their corresponding eigenvectors for a particle with constant momentum $\vec{p} = p\hat{z}$. Normalize the eigenvectors so that one of the components in the relevant energy two-spinor is unity.

(3) Explain why we expect the free particle Dirac Hamiltonian $\underline{\underline{H}} = \underline{\underline{\vec{\alpha}}} \cdot \vec{p} + \underline{\underline{\beta}} m$ to be parity symmetric. Define the parity opertor as $\underline{\underline{\mathcal{P}}} \equiv \underline{\underline{\beta}} P$ where P is the parity operator for 3D space, that is $P^\dagger \vec{r} P = -\vec{r}$. Show that $\underline{\underline{\mathcal{P}}}$ is unitary and implies that $\underline{\underline{H}}$ is parity symmetric.

(4) Three identical particles are in a one-dimensional harmonic oscillator potential well with classical angular frequency ω.

(a) Write the complete time-independent Hamiltonian and express it in coordinate space as a differential equation whose solution is the three-body wave function $\psi(x_1, x_2, x_3)$.

(b) Assume the particles have zero spin. Use the single particle wave functions to construct the ground state wave function $\psi_0(x_1, x_2, x_3)$, and show that it satisfies the differential equation in (a). Find the ground state energy.

(c) Repeat (b) assuming the particles have spin-1/2. Use a Slater determinant to construct the ground state wave function $\psi(x_1, x_2, x_3)$. Can you write the ground state as a product of separate spatial and spin parts?

(5) We showed in class that the Hamiltonian and momentum in the Klein-Gordon field, quantized in a "big box" of side length L are given by

$$H = \sum_{\vec{k}} \omega_{\vec{k}} \left(b^\dagger_{\vec{k}} b_{\vec{k}} + c^\dagger_{\vec{k}} c_{\vec{k}} + 1 \right) \qquad \text{and} \qquad \mathbf{P} = \sum_{\vec{k}} \vec{k} \left(b^\dagger_{\vec{k}} b_{\vec{k}} + c^\dagger_{\vec{k}} c_{\vec{k}} \right)$$

where the $b^\dagger_{\vec{k}}$ and $c^\dagger_{\vec{k}}$ are independent field creation and annihilation operators. Using the definition we created for $b^\dagger_{\vec{k}}$ and $c^\dagger_{\vec{k}}$, show that the integrated "charge" given by

$$Q = \int d^3r\, j^0(\vec{r}, t) \qquad \text{where} \qquad j^\mu(\vec{r}, t) = -iC \times \left[\psi(\partial^\mu \psi^\dagger) - \psi^\dagger(\partial^\mu \psi) \right]$$

where C is an arbitrary constant, can be written as

$$Q = C \sum_{\mathbf{k}} \left[b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - c^\dagger_{\mathbf{k}} c_{\mathbf{k}} \right]$$