PHYS4702 Intro Quantum Mechanics II HW#8 Due 22 Oct 2024

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Write the time independent Schrödinger equation in one dimension x for an infinite square well with $-a \leq x \leq a$. Apply boundary conditions at $x = \pm a$ and find the solution. (You'll be led to having to find the determinant of a two dimensional matrix.) Show that the solutions naturally divide into categories of even or odd parity.

(2) Write the time independent Schrödinger equation in three dimensions x, y, and z for the isotropic harmonic oscillator potential

$$V = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) = \frac{1}{2}m\omega^2 r^2$$

Recall the solution based on the one-dimensional harmonic oscillator, and show that the parity of the 3D solution is $(-1)^N$ for the energy eigenvalues $E_N = (N + 3/2)\hbar\omega$.

(3) Write down the parity transformation for $\vec{r} \to -\vec{r}$ in terms of spherical coordinates (r, θ, ϕ). Then prove that the parity of a wave function $\psi(r, \theta, \phi) = R(r)Y_l^m(\theta, \phi)$ is $(-1)^l$. Prove this with whatever definition you like for the spherical harmonics for $m \geq 0$. Note that the spherical harmonics for $m < 0$ are defined by

$$Y_l^{-m}(\theta, \phi) = (-1)^m \left[Y_l^m(\theta, \phi)\right]^*$$

(4) A state $|\alpha\rangle$ is shown to be in a simultaneous eigenstate of two Hermitian operators A and B for which $AB + BA = 0$. (We say that A and B *anti*-commute.) Derive a simple relationship between the eigenvalues of A and B. Illustrate what you discover using the operators momentum $\vec{p}$ and parity $\mathcal{P}$.

(5) A particle of mass m and energy $E > 0$ moves in one dimension x through an infinite series of equally-spaced δ-function potentials. Parameterize the potential as follows:

$$V(x) = \sum_{n=-\infty}^{\infty} \left(\frac{\hbar^2}{2m}\lambda\right) \delta(x - na)$$

(a) Derive a relationship between the derivatives of the wave function $\psi(x)$ near $x = 0$ and the value $\psi(0)$. Recall Problem 4 from Homework #5 last semester.

(b) Write $\psi(x) = Ae^{ik'x} + Be^{-ik'x}$ for $-a < x < 0$ where $E = \hbar^2 k'^2/2m$. Use Bloch's Theorem in the form $\psi(x + a) = e^{ika}\psi(x)$ for some real parameter k to derive an expression for $\psi(x)$ in the region $0 < x < a$.

(c) Using part (a) and the continuity of the wave function at $x = 0$, derive

$$\lambda \sin k'a + 2k' \cos k'a = 2k' \cos ka$$

(d) Show that this equation implies that energy eigenvalues can only exist within continuous "bands." Set $\lambda a = 20$ and derive numerical values for the upper and lower limits of $k'a$ for the first few energy bands.