PHYS4702 Intro Quantum Mechanics II HW#6 Due 8 Oct 2024

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Form the matrix representation of the operator $2\vec{L}\cdot\vec{S}$ in the basis $|1\rangle \equiv |l, m_l; +\hat{\mathbf{z}}\rangle$ and $|2\rangle \equiv |l, m_l+1; -\hat{\mathbf{z}}\rangle$ and show that the eigenvalues $j(j+1)\hbar^2$ of $\vec{J}^2$, where $\vec{J} = \vec{L} + \vec{S}$, are given by $j = l \pm 1/2$.

(2) Use your results from problem **(1)** above to show that

$$\begin{aligned} \left| j = l + \frac{1}{2} \right\rangle &= \sqrt{\frac{l+m_l+1}{2l+1}}\,|l, m_l; +\hat{\mathbf{z}}\rangle + \sqrt{\frac{l-m_l}{2l+1}}\,|l, m_l+1; -\hat{\mathbf{z}}\rangle \\ \text{and} \qquad \left| j = l - \frac{1}{2} \right\rangle &= \sqrt{\frac{l-m_l}{2l+1}}\,|l, m_l; +\hat{\mathbf{z}}\rangle - \sqrt{\frac{l+m_l+1}{2l+1}}\,|l, m_l+1; -\hat{\mathbf{z}}\rangle \end{aligned}$$

(3) Show that the equation for $u(r) = rR_{nl}(r)$ for the hydrogen atom can be written as

$$u'' \equiv \frac{d^2u}{dr^2} = \left[\frac{l(l+1)}{r^2} - \frac{2}{a_0 r} + \frac{1}{n^2 a_0^2}\right] u(r)$$

where a_0 is the Bohr radius. Now for some power s, and expectation values in an eigenstate $|nlm\rangle$, express $\int_0^\infty (ur^s u'')dr$ in terms of $\langle r^s\rangle$, $\langle r^{s-1}\rangle$, and $\langle r^{s-2}\rangle$. Next, reduce the u'' by integrating by parts. Then show that

$$\int_0^\infty (ur^s u')dr = -\frac{s}{2}\langle r^{s-1}\rangle \qquad \text{and} \qquad \int_0^\infty (u'r^s u')dr = -\frac{2}{s+1}\int_0^\infty (u'' r^{s+1} u')dr$$

Finally, use this to prove Kramers' relation

$$\frac{s+1}{n^2}\langle r^s\rangle - (2s+1)a_0\langle r^{s-1}\rangle + \frac{s}{4}\left[(2l+1)^2 - s^2\right] a_0^2 \langle r^{s-2}\rangle = 0$$

Demonstrate by explicit calculation that this works for $s = 2$ in the $2s$ and $2p$ states.

(4) Use your results from problem **(3)** above to show that

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{l(l+1/2)(l+1)n^3 a_0^3}$$

(5) By reviewing our derivation of the spin-orbit interaction for the hydrogen atom, show that for a general spherically symmetric electrostatic potential energy function $U(r)$ the spin-orbit perturbation can be written as

$$V = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dU}{dr}\vec{L}\cdot\vec{S}$$

This form is useful for alkali atoms like sodium, and also for understanding the structure of energy levels in the nuclear shell model.