PHYS4702 Intro Quantum Mechanics II HW#2 Due 10 Sep 2024

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) The quaternions i, j, and k are defined as $\mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -1$ and $\mathsf{ijk} = -1$. Show that

(a) $\mathsf{ijk} = \mathsf{kij} = \mathsf{jki}$
(b) $\mathsf{ij} = \mathsf{k}$, $\mathsf{ki} = \mathsf{j}$, and $\mathsf{jk} = \mathsf{i}$
(c) $\mathsf{ik} = -\mathsf{ki}$

(2) Use quaternions as $SU(2)$ matrices to rotate a 3D vector of length a in the z-direction about the x-axis by an angle ϕ, and explain why you get the answer you expected.

(3) Prove that (quantum mechanically) $\vec{L}\cdot\vec{M} = 0$ where the Runge-Lenz vector $\vec{M}$ is defined in Problem (5) of HW #1. (Hint: You can show that the two terms in $\vec{M}$ each give zero for the inner product.)

(4) This problem concerns the vector operators $\vec{N} \equiv (-m/2E)^{1/2}\vec{M}$, $\vec{I} = (\vec{L}+\vec{N})/2$, and $\vec{K} = (\vec{L}-\vec{N})/2$ that we defined in class.

(a) Inventing a fourth spatial dimension and writing $N_i = r_i p_4 - r_4 p_i$, prove that $\vec{N}$ has the same commutation relations with $\vec{L}$ that we derived in class, that is $[N_i, L_j] = i\hbar\epsilon_{ijk}N_k$ and $[N_i, N_j] = i\hbar\epsilon_{ijk}L_k$.
(b) Show that $\vec{I}$ and $\vec{K}$ satisfy the usual angular momentum commutation relations as two independent operators, that is $[I_i, I_j] = i\hbar\epsilon_{ijk}I_k$, $[K_i, K_j] = i\hbar\epsilon_{ijk}K_k$, and $[I_i, K_j] = 0$.

(5) The isotropic harmonic oscillator in 2D has the Hamiltonian

$$H = \frac{1}{2m}\vec{p}^2 + \frac{1}{2}m\omega^2\vec{r}^2 = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2+y^2\right)$$

Write this in terms of the annihilation and creation operators a_x, $a_x^\dagger$, a_y, and $a_y^\dagger$. Define the operators $K_+ = \hbar\left(a_x^\dagger a_y + a_y^\dagger a_x\right)$, $K_- = i\hbar\left(a_x^\dagger a_y - a_y^\dagger a_x\right)$, and $K_0 = \hbar\left(a_x^\dagger a_x - a_y^\dagger a_y\right)$. Then

(a) Show that $K_\pm$ and K_0 are Hermitian and find their commutators with the Hamiltonian.
(b) Show that the commutators of $K_\pm$ and K_0 with each other are the algebra of $SU(2)$.
(c) In the (two-dimensional) subspace of states with $E = 2\hbar\omega$ find the matrix representations of $K_\pm$ and K_0. What are these matrices? (You've seen them before.)