PHYS4702 Intro Quantum Mechanics II HW#1 Due 3 Sep 2024

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

The first two problems of this assignment are meant to get you back up to speed on the hydrogen atom, which we studied at the end of last semester. We'll be making more use of these wave functions in this course, so I advise you to prepare some code to calculate them. You can use a general formula, for example in MQM3e Section B.6, the series solution from last semester, or just look them up.

(1) Calculate and make plots of the (scaled) radial wave functions $a^{3/2}R_{nl}(r)$ and the radial probability densities $a^3r^2|R_{nl}(r)|^2$ as a function of r/a for the hydrogen atom for all states with $n=1$, $n=2$, and $n=3$, where a is the Bohr radius. You will find examples of such plots in many textbooks, including Figure 10.5 in Townsend.

(2) Calculate the maxima in the radial probability distributions in terms of atomic number Z and Bohr radius a for a one-electron atom in the $n=1$; $n=2$ and $l=0$; and $n=2$ and $l=1$ states. Also calculate the root-mean-square radius $\langle r^2\rangle^{1/2}$ and compare the results.

(3) We call $SU(2)$ the set of all unitary 2×2 matrices, with determinant equal to unity. We claim that $SU(2)$ is a "group" under matrix multiplication, and that it is a "representation" of spin-1/2 quantum mechanical states.

(a) Show that the identity $\underline{\underline{1}}$ and the three Pauli matrices $\underline{\underline{\vec{\sigma}}}$ times i are members of $SU(2)$.
(b) Recall how we wrote the representation of the rotation operator for spin-1/2. Use this to prove that $SU(2)$ is indeed a group under multiplication.
(c) Why do we say that the group of rotations in three dimensions is "isomorphic" to $SU(2)$, i.e. there is a one-to-one correspondence between members of the two groups.
(d) What do you get if you rotate the spin-up state $|+\hat{\mathbf{z}}\rangle$ by 2π about the z-axis?

(4) The group of all real, orthogonal 3×3 matrices with unit determinant is called $SO(3)$.

(a) Why is the group of rotations in three dimensions naturally isomorphic to $SO(3)$?
(b) Even though they are both infinite, $SU(2)$ is larger than $SO(3)$ because any element of the latter maps onto more than one element of the former. Find a couple of examples of this by comparing rotations in the two representations. BTW, mathematicians say that $SU(2)$ is a "double cover" of $SO(3)$.

(5) For a system with potential energy $V(\vec{r})=-Ze^2/r$, show that the operator

$$\vec{M}=\frac{1}{2m}(\vec{p}\times\vec{L}-\vec{L}\times\vec{p})-\frac{Ze^2}{r}\vec{r}$$

is Hermitian and represents a conserved observable. (We will use this quantity, called the Runge-Lenz vector, to find a special symmetry of the hydrogen atom which explains the degeneracy of its energy levels.) You can use results from last semester, including following Problem (2) from Homework #4 to state that $[p_i, F(\vec{r})]=-i\hbar\partial F/\partial x_i$.